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Preface

The analysis and optimization of structures and structural elements is a field of research which presumes the effective cooperation of experts from different areas including engineers, mathematicians, IT experts etc.

The aim of the 2nd International Conference “Optimization and Analysis of Structures 2013” is to stimulate and promote research and applications within applied mechanics and optimization and provide a forum for personal contacts, also for dissemination of existing knowledge in mechanics. We hope that the conference will stimulate the interchange of ideas in the theoretical and applied mechanics, solid mechanics, fracture mechanics as well as in the theory of optimization and numerical methods of optimization.

This, the 2nd International Conference OAS will take place in August, 25–27, 2013. The first conference of this series, the OAS 2011 was dedicated to the 90th jubileum of Professor Ülo Lepik.

The organizing committee gratefully acknowledge the authors of presentations for their contribution.

Jaan Lellep, Chairman of the organizing committee
Ella Puman, Scientific secretary of the conference
Institute of Mathematics, University of Tartu
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Influence of dynamic processes in a film on damage development in an adhesive base

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Abstract. In the process of deformation of multilayer structures, significant stresses can arise on the foundation-coating interface which can result in fracture or coating separation. The action of static or impact loads on damage onset and development in the adhesive layer in such multilayer structures has been investigated almost completely, but similar processes in the case of suddenly applied vibration loads have not been studied to a large extent. The latter draw attention because of the fact that even small variable actions can localize vibrations near the imperfections (inclusions, defects, etc.) and can be accompanied with an increase in the damage of the adhesion layer. In the present paper, the possibility of vibration localization in damaged regions and the influence of the localization on the damage development till the film separation are studied. Some conditions under which damage behavior is determined by the localized oscillating part of the solution are derived.

Keywords: damage, wave localization, film separation.

1. Introduction

Thin-layer coatings connected with the main structure by an intermediate thin adhesion layer are often used in contemporary structures as protective or reinforcing elements. In the process of deformation of such multilayer structures, significant stresses can arise on the foundation-coating interface because of the difference in their physical and mechanical properties, which can result in fracture or coating separation. The problems of delamination of multilayer structures under static and dynamics (impact) loads are discussed in numerous sources in the literature [1]–[6]. The action of static or impact loads on damage onset and development in the adhesive layer in such multilayer structures has been investigated almost completely, but similar processes in the case of unsteady (vibration) loads have not been studied to a large extent. The latter draw attention because of the fact that even small variable actions can localize vibrations near the imperfections (inclusions, defects, etc.) [7] and can be accompanied with an increase in the damage of the adhesion layer, which results in film delamination. In the present paper, the possibility of vibration localization in damaged regions and the influence of localization on damage development until the film separation are studied.

2. Problem formulation

Since the thickness of a thin-layer coating is usually much less than the foundation characteristic dimensions, the coating fixed to the foundation is replaced in the first approximation by a film on an elastic foundation. The first approximation is when the film
is modeled by an infinite string lying on an elastic foundation with a coefficient depending on the damage of the adhesion layer of the substrate. The elastic foundation replaces the action of the substrate and the main material on the film. The coefficient of the elastic foundation of the string is equal to the total rigidity of springs connected in series and the rigidities of the adhesion layer and the main material,

\[
\gamma u_{xx} - K(n)u - \rho_0 u_{tt} = Q(t, x), \quad x \in (-\infty, \infty), \quad t > 0,
\]

where \(u\) is the displacement, \(Q\) is the external force, and \(K(n)\) is the rigidity of the elastic foundation depending on the adhesion layer damage \(n\), the layer initial rigidity \(G_0\), and the rigidity \(k_0\) of the main material. The kinetic equation for the damage has the form

\[
\frac{\partial n}{\partial t} = \beta H(\mu(n)|u| - \Delta)(1 - n).
\]

Here \(\beta\) and \(\Delta\) are positive variables and \(H\) is the Heaviside function. The case where the external load \(Q\) has the form

\[
Q(t, n) = A \sin(\omega t) \delta_\varepsilon(x) H(t)
\]

is considered. Here \(\delta_\varepsilon\) is a smooth delta-function, which can be defined, for example, as

\[
\delta_\varepsilon(x) = \varepsilon^{-1}(2\pi)^{-1/2} \exp\left(-\frac{x^2}{2\varepsilon^2}\right).
\]

The following boundary and initial conditions are posed:

\[
\begin{align*}
    u(x, 0) &= 0, \quad u_t(x, 0) = 0, \quad x \in (-\infty, \infty), \\
    n(x, 0) &= 0, \quad x \in (-\infty, \infty).
\end{align*}
\]

It should be noted that equation (2) was derived under the assumption that \(0 \leq n(x, t) \leq 1\) for all \(n\) if \(0 \leq n(x, 0) \leq 1\) and \(t > 0\), then there exist \((x_*, t_*)\) such that \(n_t > 0\) and \(n = 1\), but this contradicts (2). Hence (2) is well defined. It is assumed that the complete fracture of the adhesion layer material, which results in the separation of part of the film, occurs at those points \(x\) and at the time \(t\) at which the damage \(n\) attains a certain critical level \(n_*\).

3. Asymptotic solution for small \(\beta\)

Consider the case where \(\beta\) can be treated as a small parameter, i.e., the case where the damage rate is small compared with the speed of the wave propagation in the string with undamaged foundation. Then \(\partial n/\partial t \ll 1\) and \(K\) is a function of the slow time \(\tau = \beta t\). In what follows, the multiple scale method is used and the solution of (1) is sought in the form

\[
u = U_0(x, t, \tau) + \beta U_1(x, t, \tau) + \ldots
\]

By substituting this expression into the main equation (1), one obtains

\[
U_{0xx} - K(n)U_0 - \rho_0 U_{0tt} = Q(t, x), \quad x \in (-\infty, +\infty), \quad t > 0.
\]

Here we have taken into account the fact that \(n = n(x, \tau)\) can be viewed as a parameter.
In the first approximation, the damage has the form
\[
\frac{\partial n}{\partial \tau} = H \left( \frac{k_0 U_0(x,t,\tau)}{k_0 + G(n)} - \Delta \right), \quad \tau = \beta t.
\]
The damage begins to grow at the points where the deformation criterion is satisfied,
\[
k_0 U_0(x,t,\tau) \frac{1}{k_0 + G_0[1 - n(x,\tau)]} = \Delta.
\] (3)

The time \( T_{beg}(x) \), which is a solution of (3), can be called the time of beginning of damage growth. The correction to the solution \( U_1 \) can be obtained from the equation
\[
\gamma U_{1xx} - K(n) U_1 - \rho_0 U_{1tt} = Q_1(t,x,U_0), \quad x \in (-\infty, +\infty), \quad t > 0, \quad Q_1 = 2\rho_0 U_{0tt}.
\]

Consider the case where \( k_0 \ll 1 \) and \( G_0 = O(1) \). Let us introduce the quantity
\[
\bar{k}_0 = \frac{k_0 G_0}{k_0 + G_0}.
\]

Then, using \( \bar{k}_0 \) as a small parameter, one can obtain the following approximate solution for \( u \):
\[
u = U_0 + \bar{k}_0 U_1.
\]
Here \( U_0 \) is the principal term of the solution, and \( \bar{k}_0 U_1 \) is a correction. The principal term \( U_0 \) has the form
\[
U_0 = W + \bar{V}(x) \sin(\omega t),
\]
where \( \bar{V}(x) \) satisfies the equation
\[
\gamma \bar{V}_{xx} - \bar{k}_0 \bar{V} - \rho_0 \omega^2 \bar{V} = \delta(x).
\] (4)

As a result, one has
\[
V(x) = (2\alpha)^{-1} \exp(-\alpha|x|), \quad a = \sqrt{\frac{\omega^2 \rho_0 + \bar{k}_0}{\gamma}}.
\]

To obtain \( W \), one should solve the Cauchy problem
\[
\gamma W_{xx} - k_0 W - \rho_0 W_{tt} = 0, \quad W(x,0) = 0, \quad W_t(x,0) = \omega V(x).
\] (5)

Note that the left-hand side of the equation formally contains the term \( \bar{k}_0 W \). This term was introduced to suppress the secular terms that may appear if the pure wave operator is used on the left-hand side in (5). This problem can be solved by the Fourier method. Let \( V_k \) denote the Fourier coefficients of \( V(x) \). In what follows, the exact form of these coefficients is of no importance. Then one has
\[
W(x,t) = \omega \int_{-\infty}^{+\infty} V_k \omega_k^{-1} \sin(\omega_k z)dk, \quad \omega_k = \rho_0^{-\frac{1}{2}} \sqrt{\gamma k^2 + \bar{k}_0},
\] (6)
and for \( t \ll k_0^{-1} \) one can obtain the following asymptotics for \( W \):
\[
W(x,t) = \omega (2c)^{-1} [\bar{W}(x - ct) - \bar{W}(x + ct)],
\]
\[
\frac{d\bar{W}(z)}{dz} = \bar{V}(z), \quad z = x - ct.
\]
where \( \bar{V} \) is determined by (4). Here \( c \) is the speed of the traveling wave, \( c^2 = \gamma/\rho_0 \). The quantity \( U_0 \) is the sum of the following two terms: the function \( \bar{V} \) is localized near \( x = 0 \), while the expression \( \bar{W}(x - ct) - \bar{W}(x + ct) \) presents two waves of kinetic type that
move in opposite directions. So the first term describes the wave part of the solution, while the second term has the character of localized oscillations. For $t \gg k_0^{-1}$, one can use the stationary phase method to estimate the integral on the right-hand side in relation (6). The standard computation of the integral in (6) showed that the contribution of the wave part $\bar{W}$ decreases for large times,

$$\bar{W} = O((k_0^{-1}t)^{1/2}).$$

To find the correction $U_1$, one should solve the wave equation with the right-hand side

$$\gamma U_{1xx} - \bar{k}_0 U_1 - \rho_0 U_{1tt} = \bar{k}_0 \Phi(x, \tau) U_0 + 2\rho_0 U_{0tt},$$

where

$$\Phi = \frac{k_0^2 G_0 n}{(k_0 + G_0) [(k_0 + G_0)(1 - n)]]}.$$

By solving this equation by the Fourier method, one can show that the correction $U_1$ is bounded as $t \to +\infty$, and hence this correction does not give a significant contribution to the asymptotics of $T_{beg}(x)$.

4. The case of $\beta$ and $K(n) \gg 1$

In this case, one can find the asymptotics for $\bar{u} = u - V(x) \sin(\omega t)$. Neglecting the small term $\gamma \bar{u}_{xx}$, one obtains

$$K(n) \bar{u} + \rho_0 \bar{u}_{tt} = 0, \quad \bar{u}(x, 0)_t = \omega V(x), \quad \bar{u}(x, 0) = 0.$$

Then the expression for $\bar{u}(x, t)$ has the form

$$\bar{u}(x, t) = \omega \rho_0^2 K^{1/2} (n(x, 0)) V(x) \sin \left\{ \frac{\gamma}{\rho_0^2} \int_0^t K^{1/2} (n(x, s)) ds \right\}.$$

As a result, we obtain

$$u(x, t) = V(x, t) \sin(\omega t) + \omega \rho_0^2 K^{1/2} (n(x, 0)) V(x) \sin \left\{ \frac{\gamma}{\rho_0^2} \int_0^t K^{1/2} (n(x, s)) ds \right\}.$$

This expression has an oscillating character and does not contain any wave contribution.

5. Determining the time $T_{beg}$ of beginning of damage growth

If the film is loaded by an external force, then the initial damage of the substrate varies in space and time according to the proposed the damage growth law. Assume that the beginning of the damage growth is determined by a deformation criterion, namely, by the condition that the string displacement exceeds a certain prescribed value $\Delta$. Assume also that $n_0(x, t) < n_*$ and introduce the quantity $\varphi(x, t)$ as

$$\varphi(x, t) = k_0 U(x, t) \{k_0 + G_0[1 - n_0(x)]\},$$

where $U$ is given by the equation

$$\gamma U_{xx} - K(n_0(x)) U - \rho_0 U_{tt} = Q(x, t).$$

Since $T_{beg}$ is small compared with time divided by $\beta^{-1}$, one can replace $n(x, t)$ in the above equations by $n_0(x)$. Assume that, for some $x$, the expression for $\varphi(x, n_0, t)$ can take the value $\Delta$ at some instant of time $T_{beg}(x)$:

$$\varphi(x, T_{beg}(x)) = \Delta.$$
In this case, for $K(n) \ll 1$, one obtains the following equation for $T_{\text{beg}}(x)$:

$$Ak_0 \sin(\omega T_{\text{beg}}) \frac{\exp\{-a(n_0(x))|x|\}}{2a(n_0(x))} = \{k_0 + G_0[1 - n_0(x)]\} \Delta.$$  

It is clear that $T_{\text{beg}}(x)$ is minimal at the point $x = 0$. Let $n_0(0) = \bar{n}_0$, and introduce the following threshold value:

$$M(n, x) = 2A^{-1}k_0^{-1}\exp\{a(n_0(x))\}\{k_0 + G[1 - n_0(x)]\}\Delta(\bar{n}_0).$$

Then

$$T_{\text{beg}} = \omega^{-1} \arcsin M(n, x), \quad M(n, x) \leq 1, \quad T_{\text{beg}} = +\infty, \quad M(n, x) > 1.$$  

For large $\omega$, one can use a simplified asymptotic relation for $a$,

$$a = \left(\frac{\rho_0}{\gamma}\right)^{\frac{1}{2}} \omega = c^{-1} \omega,$$

where $c$ is the wave propagation speed.

6. Determining the motion of the boundary $L(t)$ of the damage region.

The wave and localized fracture modes

Assume that the growth of $n$ is monotone in time. Then one can obtain the following solution of equation (2):

$$n(x, t) = 1 + [n_0(x) - 1] \exp\{-\beta[t - T_{\text{beg}}(x)]\}.$$  

Taking into account this solution, one can obtain the following equation for determining the position of the front $L(t)$ of the damage region:

$$n_\ast = 1 + [n_0(L) - 1] \exp\{-\beta(t - T_{\text{beg}}(L))\}.$$ (7)

Under the condition $\omega^{-1} \gg \beta$, equation (7) can be reduced to a simple equation for $L(t)$. Indeed, in this case, regardless of the initial data, one has $T_{\text{beg}} \ll \beta^{-1}$. Therefore, Equation (7) can be replaced by $n_\ast = 1 + [n_0(L) - 1] \exp(-\beta t)$, which implies

$$1 + (n_\ast - 1) \exp(\beta t) = n_0(L(t)).$$ (8)

Note that, for example, for the Gaussian distribution of the initial damage $n_0$ (and for all distributions $n_0(|x|)$ decreasing with the coordinate), there are two fronts (boundaries), namely $L$ and $-L$ (where $L(t)$ increases with time). The growth of $L$ is bounded; i.e., (8) does not have a solution for large $t$. In the general case of nonmonotone growth of the damage region, $L$ can be determined numerically by using the asymptotic solution obtained above. Thus, an approximate equation was obtained for a slowly monotonically propagating front of the damage region, and this equation depends on the initial distribution of the damage and is independent of the initial value $u(x, t)$ and the force $Q(x)$.

The nonmonotone dynamics of $L$ is mainly described by a stepwise function.
7. Resonance effect

Now it is assumed that \( n_0(t) \) is a well localized function. In this case, \( K(n(x, t))u(x, t) \) can be approximated by the quantity \( \bar{K}(t)u(0, t)\delta(x) \), where \( \bar{K}(t) \) is the coefficient proportional to \( K(n(0, t)) \). This coefficient is determined by the relation

\[
\int_{-\infty}^{\infty} K(n(x, t)) \, dx = \bar{K}(t).
\]

Note that \( \bar{K}(t) \) is a slow function of \( t \), because \( d\bar{K}/dt \leq \beta \). Then one has

\[
u_{xx} - c^2u_{tt} = \gamma^{-1}[A + \bar{K}(t)u(0, t)]\delta(x).
\]

As before, the solution is sought in the form \( u = \sin(\omega t)V(x) \). For \( V \), one has

\[
V_{xx} - c^2\omega^2V = \gamma^{-1}[A + \bar{K}(t)V(0)]\delta(x),
\]

and hence

\[
V(x) = 2a\gamma^{-1}[A + \bar{K}(t)V(0)] \exp\{-a(n(\tau, 0))[x]\}.
\]

This expression implies the following expression for the slowly varying amplitude \( V(0, \tau) \):

\[
V(0, \tau) = A[2a(n(\tau, 0))\gamma - \bar{K}(n(\tau))]^{-1}, \quad \tau = \beta t.
\]

This expression shows that resonance may occur if \( \bar{K}(t) \) is close to \( 2a\gamma \). Since \( \bar{K}(t) \) decreases with increasing \( t \), this effect is possible only if \( \bar{K}(0) > 2a\gamma \) at the initial time instant. This effect is also possible only for the frequencies \( \omega < \omega_0 \). Note that this conclusion is based on the fact that \( a(n) \) also decreases with increasing \( n \), while \( a(n) \) decreases with \( n \) slower than \( K \).

8. Conclusion

First, consider how \( n(x, t) \) grows in the case where \( k_0 \) is a small variable. The following three scenarios of the damage function behavior are possible. These scenarios can be described by the expression obtained above for the film displacement in the form \( u = U_0 + k_0U_1 \), where \( U_0 \) is the principal term of the solution and \( k_0U_1 \) is a correction. The quantity \( U_0 \) is determined by the expression

\[
U_0(x, t) = A\left\{ \frac{\omega}{2c} [W(x - ct) - W(x + ct)] + V(x) \sin(\omega t) \right\},
\]

where \( V(x) \) satisfies the relation

\[
\gamma V_{xx} - K(n(x, t))V - \rho_0\omega^2V = \delta(x)
\]

and \( W(x) = V(z) \). Let us introduce the notation

\[
\Delta_1(n) = \Delta k_0^{-1}[k_0 + G_0(1 - n)].
\]

Here \( \gamma = 1 \). The first possible scenario of damage region behavior is a monotone growth of \( n \). The damage \( n \) increases if \( U_0 > \Delta_1(n) \). The monotone growth is possible if this inequality is satisfied for all \( t \). Note that the oscillating term \( V \) is localized near zero and has amplitude \( A_V = 2a^{-1}(n) \) at this point. The wave term for \( x \) close to zero has amplitude \( A_W = \omega[2ca(n(\tau, 0))^2]^{-1}; \) i.e., \( A_V \approx A_W \exp\{-a(n)[x]\} \). Note that these amplitudes \( A_W \) and \( A_V \) depend on \( x \). For example, if \( A_W - A_V > \Delta_1(n) \), then the damage will grow monotonically. Indeed, then \( A_W + A_V \sin(\omega t) > \Delta_1(n) \) for all \( t \).
The second scenario of damage behavior is the stepwise growth of \( n \). In this case, the damage increases on some time intervals and is constant between these intervals. Such a scenario is possible if \( A_W \) and \( A_V \) are of the same order of magnitude. This second scenario can be transformed into the first one if \( n \) becomes sufficiently large, because \( \Delta_1(n) \) decreases with increasing \( n \). The third scenario is that the damage does not increase. This scenario is possible if \( A_W + A_V < \Delta_1(n_0) \). This condition can be satisfied for sufficiently large frequencies \( \omega \). Note that the first and third scenarios take place for large \( x \). Moreover, since \( \Delta_1(n) \) decreases with increasing \( n \), the time intervals where \( n = \text{const} \) in the second scenario decrease in time and can even vanish in the end. If \( k_0 \) is not small, then the damage behavior is determined by the localized oscillating part of the solution and the wave part of the solution plays a key role on large times.

References

Accuracy analysis of vibration-based identification of elastic parameters

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Abstract. Proposed is the method for the analysis of parameter uncertainty effects on the accuracy of vibration-based elastic parameter identification. The method uses the approach of minimization of the discrepancy between physically measured and numerically calculated natural frequencies. The Monte Carlo simulation is used for obtaining correlations between frequency deviations. These correlation coefficients and deviations of physical frequency measurements are used for confidence interval calculations for identified parameters. The method gives the possibility of optimal choice of frequency sets and weighting coefficients for identification of each identifiable parameter.

Keywords: identification of mechanical properties, vibration, metamodeling, composites.

1. Introduction

The method of identification of elastic parameters (Young's moduli, shear moduli, Poisson's ratio) using eigenfrequency measurements of specimens is very old [4]. Currently there is an extensive amount of literature on the identification of elastic properties of layered composite materials using physical measurements and numerical calculations of natural frequencies, mostly using Finite Element Method (FEM) [5], [6], [7], [8], [9]. The traditional numerical-experimental identification procedure is based on the minimization of discrepancy between numerical and experimental results [5], [7], [8]. During the first years of using this method, the main problem was the minimization of the discrepancy functional. Today, using modern numerical experimental designs and nonparametric approximation methods, the discrepancy minimization is not a great problem. However, the estimation of the variance of obtained identified parameters is a pressing problem. The errors of identification depend on errors introduced by material production, cutting testing specimens, physical measurement errors and errors caused by disregarding significant factors in the finite element model. A large amount of literature is devoted to analysis of the accuracy of FEM, but the influence of errors of physical experiments, caused by parameter variance during material production, specimen preparation and errors of registration and measurement of natural frequencies, is significantly less studied.

Here we propose a method of variance calculation using Monte Carlo simulation of geometrical and physical uncertainty, but at the beginning the main idea of vibration-based identification [5], [7], [8] will be explained.

2. Vibration-based identification of elastic parameters

Identifiable elastic parameters usually are Young’s moduli, shear moduli, Poisson’s ratio. The classical idea is to find those values of elastic parameters of the mathematical FEM model which will give minimal discrepancy between calculated and physically measured eigenfrequencies.
We will designate the vector-column of \( n \) physically measured natural frequencies as \( f^{\text{EXP}} \):

\[
f^{\text{EXP}} = \begin{bmatrix} f_1^{\text{EXP}} & f_2^{\text{EXP}} & \ldots & f_n^{\text{EXP}} \end{bmatrix}^T,
\]

(1)

vector-column of numerically calculated (mostly by FEM) eigenfrequencies as \( f^{\text{FEM}} \):

\[
f^{\text{FEM}} = \begin{bmatrix} f_1^{\text{FEM}} & f_2^{\text{FEM}} & \ldots & f_n^{\text{FEM}} \end{bmatrix}^T,
\]

(2)

and vector-column of \( m \) physically identifiable parameters of elasticity as vector-column \( E \):

\[
E = [E_1, E_2, \ldots, E_m]^T.
\]

(3)

A superscript \( T \) denotes the matrix transpose operation. The number \( m \) of identified parameters can be different, including elastic modulus and Poisson’s ratio for different composite layers. The discrepancy \( \Phi \) between measured and calculated natural frequencies is measured as weighted sum of \( n \) differences

\[
\Phi = \sum_{i=1}^{n} W_i |f_i^{\text{EXP}} - f_i^{\text{FEM}}|^l,
\]

(4)

where \( W_i \) – nonnegative weighting coefficient for \( i \)-th frequency, \( l \) – positive exponent. The frequently used weighting method for discrepancy measure is squared relative error

\[
\Phi = \sum_{i=1}^{n} \left( \frac{f_i^{\text{EXP}} - f_i^{\text{FEM}}}{f_i^{\text{EXP}}} \right)^2.
\]

(5)

The discrepancy minimization approach means that input parameters which give minimal value of functional \( \Phi \) will be considered as identified values for unknown parameters \( E \):

\[
E^* = \text{arg min}_E \Phi(E).
\]

(6)

When using FEM software, the minimization needs physical experiments as well as numerical experiments; therefore this approach is called Mixed Numerical-Experimental Technique (MNET). The flowchart of this method is shown in Fig. 2.

The traditional MNET steps are:

- **Step 1.** Preparation of specimen samples, providing frequency measurements by resonance measurements or Fourier analysis of free oscillations registered after initial excitation.
- **Step 2.** Design of numerical experiments for FEM software. The variable input factors for eigenfrequency calculations are identifiable elastic parameters. Mostly the Latin Hypercube (LH) type designs are used. Here we use LHs optimized according to Mean Square Error space-filling criterion, introduced in [1]. The values for other input parameters (geometrical, mass, density, layer configuration and others must correspond with specimens used in physical experiments.

The number of runs for numerical experiments depends on the number \( m \) of the identifiable parameters. For relatively simple plate-type specimens the calculations are fast enough to execute 100 – 300 trial runs in 15 minutes of computing time.

- **Step 3.** Executing the numerical experiments, registering and grouping the eigenfrequencies according to vibration modes.

- **Step 4.** Building the metamodel (surrogate model) for the dependency of calculated eigenfrequencies \( f^{\text{FEM}} \) on the input parameters \( E \):

\[
f^{\text{FEM}} = \hat{f}(E),
\]

(7)
where the “hat” above a function symbol means approximation. The software EDAOpt [1], created at Institute of Mechanics of Riga Technical University, was used for design of computer experiments, metamodel building and minimization of discrepancy functional. Practice shows that almost in all cases third order polynomials give the approximation of frequencies with error less than 0.01%. For some more complicated specimens the best accuracy can be obtained using nonparametric approximation methods: kriging, locally weighted polynomials.

Step 5. Finding the values of identifiable parameters by minimization of approximated discrepancy functional

\[ E^* = \arg \min_E \sum_{i=1}^{n} w_i |f_i(E) - f_i^{FEM}|^l. \]  

(8)

The software EDAOpt uses modified multi-start simulated annealing method [1] and always gives global minimums of the discrepancy functional. It must be noted that the metamodels are relatively simple for evaluation of the objective function (8) and even several millions of evaluations needs only a few minutes of computing time.

Step 6. Traditionally, the next step is the recalculation of the metamodel in the sub-area near to the found values of identified parameter values, and analysis of the significance of different elasticity parameters on natural frequencies [5], [8]. However, this analysis gives insufficient information for the estimation of accuracy of identified values. Therefore in the present work the method for accuracy estimation will be proposed.

3. Estimation of the identification errors

The errors of identification depend on errors introduced by material production, cutting of testing specimens, physical measurement errors and errors of the finite element model. Practice shows that the mode measurements using the PSV–400 vibrometer have very high accuracy. The repeated measurements for a given specimen give almost the same results. At the same time the measurements of 3–6 different specimens from the experimental sample give the estimate standard deviations from the mean from about 0.5 % up to 2 %. This means that the variance of parameters introduced by production (elasticity, density, thickness uniformity, etc.) and errors introduced by the preparation of the sample (geometrical errors, errors of density and weight estimation, microdamages created by sample cutting) have determining influence on the variance of identified elastic parameters.

Lot Many works have been devoted to the analysis of impact of elastic parameters on eigenfrequencies using derivative matrices [3], [5], [8]. In [5] the sensitivity of Poisson’s ratio determination on the aspect ratio of a rectangular plate is studied. But it hasn’t been taken into account that all frequencies are highly correlated, have different measurement errors and significance for parameter estimation.

Below is described the use of Monte Carlo simulation for estimation of identification errors.

After finding numerical values of identified parameters \( E^* \) using the discrepancy minimization method, the program EDAOpt gives the possibility of linearization of the metamodel (obtained with polynomial or kriging approximations) in the neighborhood of point \( E^* \) in the \( m \)-dimensional parameter space, obtaining the linear metamodel:

\[ \hat{f}(E) = AE + B, \]  

(9)
where \( A \) – constant matrix \( n \times m \), \( B \) – constant vector-column with \( n \) components.

Weighted discrepancy functional between calculated frequencies \( \hat{f} \) and measured frequencies \( f^{\text{EXP}} \) in the matrix form (exponent \( l = 2 \)):

\[
\Phi = (AE + B - f^{\text{EXP}})^T W (AE + B - f^{\text{EXP}}),
\]

where \( W \) is the weighting matrix, for example, squared inverse of measured frequencies:

\[
W = \text{diag} \left( \left( \frac{1}{f_1^{\text{EXP}}} \right)^2, \left( \frac{1}{f_2^{\text{EXP}}} \right)^2, \ldots, \left( \frac{1}{f_n^{\text{EXP}}} \right)^2 \right).
\]

The discrepancy minimization approach gives following:

\[
E^* = \arg \min E \Phi(E)
\]

and after equating derivatives \( \partial \Phi / \partial E_i \) to zero, we obtain the system of linear algebraic equations

\[
A^T W A E^* + A^T W (B - f^{\text{EXP}}) = 0
\]

The values of parameters which give minimal discrepancy are therefore vector-column \( E^* \)

\[
E^* = S f - H,
\]

where

\[
S = (A^T W A)^{-1} A^T W
\]

and constant vector-column

\[
H = (A^T W A)^{-1} A^T W (B - f^{\text{EXP}}).
\]

From the physical experiments with sample size \( k \) (number of specimens in the set of measurements), the mean \( \bar{f}_i \) and standard deviation of \( i \)-th frequency can be calculated [2] as

\[
\bar{f}_i = \frac{1}{k} \sum_{j=1}^{k} f_i^{\text{EXP}(j)},
\]

\[
\sigma_i = \sqrt{\frac{\sum_{j=1}^{k} (\bar{f}_i - f_i^{\text{EXP}(j)})^2}{k - 1}}.
\]

As usual, we can assume that measured means are random variables and have normal probability distribution. Then the standard deviations of frequency measurements can be used for the calculation of the standard deviation using the linear dependence between elasticity parameters and natural frequency. The problem is that measured frequencies may not be considered as independent random variables but are highly correlated with each other.

The Pearson correlation coefficients can be calculated using the set of measured frequency values, but the number of measurements usually is too small to calculate the correlation matrix between the 15 – 20 frequencies used in identification. Therefore we propose the following Monte Carlo simulation approach for the calculation of frequency correlation coefficients. The scattering of natural frequency values depends mainly on geometrical errors of the specimen and actual variance of elastic parameters and
density of material. For the rectangular plate type specimen we used the Monte Carlo method for the simulation of the influence of following geometric parameter variance:
6 error parameters of specimen cutting (plate type specimens are not exactly rectangular) see Fig. 1a, and 4 parameters describing the variation of plate thickness, see Fig. 1b:

\[ \Delta \]

\[ \phi \]

\[ R \]

Fig. 1. The parameters of geometrical deviation.

parameter \( \phi \) describing the angle between fiber orientation and edge of rectangle, parameter \( R \) - surface curvature of the plate, see Fig. 1c.

\[ \Delta \]

\[ \Delta \]

\[ \Delta \]

\[ \Delta \]

Fig. 2. Flowchart of parameter identification with variance estimation.

In addition to geometrical deviations of parameters 4 values of standard deviation for elastic parameters \((E_x, E_y, G_{xy}, \nu)\) were used for Monte Carlo simulation. All these parameters were simulated as random variables with normal density distribution and given standard deviations. The mean values for first 13 error parameters are zero, the mean values of elastic parameters and their deviations are taken from previous experiments, but can be iteratively repeated after identification.
The Monte Carlo simulation consists of $N = 2000$ calculations of eigenfrequencies using randomly generated 17 input variables. The results are $N$ values for each of first $m$ frequencies $f_{ij}$, $i = 1, ..., N, j = 1, ..., m$. The Pearson product-moment correlation coefficient between $i$-th and $j$-th frequency is

$$
\rho_{ij} = \frac{\sum_{k=1}^{N}(f_{ki} - \bar{f}_i)(f_{kj} - \bar{f}_j)}{\sqrt{\sum_{k=1}^{N}(f_{ki} - \bar{f}_i)^2} \sqrt{\sum_{k=1}^{N}(f_{kj} - \bar{f}_j)^2}}.
$$

(19)

Then, according to expression (14) the standard deviation of $k$-th identifiable parameter can be calculated according to statistical law for correlated random variables [2].

$$
STD(E^*_k) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} S_{ki}S_{kj} \rho_{ij} \sigma_i \sigma_j}, \ k = 1, ..., m,
$$

(20)

where $S_{kj}$ ($k = 1, ..., m, j = 1, ..., n$) are elements of matrix $S$, see (15), $\rho_{ij}$ is the correlation coefficient between $i$-th and $j$-th natural frequency and $\rho_i$ is the sample standard deviation of $i$-th measured frequency ($i, j = 1, ..., n$). The obtained values of standard deviations can be used for calculating confidence intervals of identified parameters.

4. Example of identification results

The proposed method was used for the identification of elastic properties of weft knitted GF/PP textile composite [10]. In our experiments with knitted GF/PP textile composite plates the PSV – 400 vibrometer was employed for non-contact measurement, visualization and analysis of structural vibrations. The flowchart of identification algorithm with accuracy estimation is shown in Fig. 2. The mean relative standard deviation of physical frequency measurements was about 0.5 – 2%. The values of correlation coefficient for first 10 frequencies vary from 0.07 (between first and second mode) up to 0.95 (between 3-th and 8-th mode).

The obtained values for $E_x, E_y, G_{xy}$ have about 2 – 3 % error with confidence level 95 %. At the same time the error of obtained Poisson’s ratio is about 12 – 15 %. The choice of different weighting coefficients in the discrepancy function (4) is very important for the accuracy improvement. For example, using frequencies 2,3,5,6,7,8 with weighting exponent $l = 2$ and $w_i = (f_i^{\text{EXP}})^{-2}$ gives 20.4 % error for Poisson’s ratio (with traditional two sigma 95 % confidence level), but the same set of frequencies with weighting coefficients $w_i = (f_i^{\text{EXP}})^{-1}$ gives 15 % error for $\nu$.

Conclusions

The presented uncertainty analysis algorithm estimates the uncertainty of the identified elastic parameters from the uncertainties of the experimentally measured quantities and uncertainties of geometrical and physical parameters of experimental specimens. This method gives the possibility of estimation of confidence intervals of determined parameters with a given confidence level. The method gives the possibility of optimal
choice of natural frequency set and weighting coefficients obtaining the best identification accuracy for each identifiable parameter separately.

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References
Buckling of a spherical segment under the flat base load

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Abstract. The analysis of stress-strain states of soft and almost soft shells under internal pressure and flat base load are important in study of intraocular pressure, which is an important characteristic in ophthalmology. In the Maklakoff method of tonometry a human eye is deformed by flat base load. The diameter of the contact zone with cornea is measured and the measured diameter length is used in estimating the intraocular pressure. When the intraocular pressure is not very high and the thickness of an eye shell (cornea) is small (for example, after refractive surgery) the cornea may buckle and detach from tonometer. It leads to errors in estimates of intraocular pressure. Shells deformations under the flat base load are large and therefore described by equations of geometrically nonlinear theory of shells. In our study the Palii-Spiro theory of anisotropic shells of moderate thickness was applied. The problem was solved by means of the method of consequent loading (delta method). Since only linear physical relations are used in delta method one can trace each step of solution of linear systems with constant coefficients. The results obtained with i) linearized non-linear equilibrium equations, ii) the method of minimization of shell elastic potential and iii) finite element method are compared.

The research leads us to the following conclusions: The flatter form of the shell makes larger the radius of the contact area between the cornea and the load and makes larger the radius of the inner unloaded area. Decreasing of the shell thickness also leads to increasing of the radius of the contact area, however, this parameter has affect less on the measurement of the intraocular pressure, than the parameter of flatness.

Keywords: intraocular pressure, deformation of shells, method of consequent loading.

1. Introduction

To measure the intraocular pressure (IOP) A. N. Maklakov in 1884 proposed to apply the flat-bottomed load (usually \(10 \, g\)) to the cornea, that eventually deforms under this load. (Fig. 1).

Fig. 1. Applanation tonometry.
Measuring the IOP one estimates the diameter of a circular contact area of the cornea and the tonometer. Each tonometer is supplied with special tables which are lately used to estimate the IOP corresponding to the measured value of the diameter. In 1920–1930, when methods of the measurement of the IOP were developed, the analysis of stress-strain state in shells was a difficult problem. It explains why nowadays the calculating of such tables is based on the empirical clinic values of the IOP. It was assumed that the elastic properties of the corneas are the same for all patients. But for example after refractive surgery cornea became thinner and cornea could lose of stability and detach from tonometer. It can lead to error in estimation of IOP on the corresponding tables. Standardization of IOP measurements, determination of its normal and pathology indexes and examination of the effect of different parameters of the eyeball onto the value of IOP are among urgent questions of today.

2. Problem formulation

We consider cornea as a spherical shallow segment of constant thickness $h$, of radius $R$ ($h \ll R$) under the inner pressure (intraocular pressure) and load with flat base. The deformations of shells in this case are large and are described by the equations of geometrically nonlinear theory of shells. According to the data obtained by E. Iomdina [1] cornea is close to transversal-isotropic shell, which has a modulus of elasticity towards the cornea thickness, two order of magnitude smaller than tangential modulus. That is why one of nonclassical theories of shells – theory of the anisotropic shells by Paliy-Spiro is used [2]. This is the theory of shells of moderate thickness. The task was solved under the assumption of axial symmetry so all stresses and deformations does not depend on circumferential coordinate. The basic kinematic assumptions of the Paliy-Spiro theory are

1) a rectilinear element normal to the middle surface of a shell remains rectilinear after deformation;

2) cosine of the slope angle of a rectilinear element normal to the middle surface is assumed to be equal to the mean transverse shear angle.

Expressed in mathematical terms, these hypotheses give

\[
\begin{align*}
  u_1 &= u + \varphi \cdot z, \quad \varphi = \varphi_0(\alpha_1) + \gamma_1, \\
  u_3 &= w + F(\alpha_1, z), \\
  \varphi_0 &= -\frac{1}{R} \frac{\partial w}{\partial \alpha_1} + \frac{u}{R},
\end{align*}
\]

where $\varphi$ are rotations of rectilinear fibers about paralele $\alpha_1$; $\varphi_0, \gamma_1$ are rotation and shear angles of normals to the middle surface in $(\alpha_1, z)$ plane, $F(\alpha_1, z)$ is a function, equal to zero at $z = 0$, $R$ is radius of spherical shell (cornea), $u$ is deflection of the middle surface along the $\alpha_1$, $w$ is normal deflection of the middle surface.

These assumptions, nonlinear relations for deformations and generalized Hook’s law allow to obtain three differential nonlinear equations of the 6-th order on functions $u(\alpha_1), w(\alpha_1), \gamma_1(\alpha_1)$ [2], [3].

The edge of the cornea is supposed to be completely fixed

\[
  u(\alpha_{10}) = 0, \quad w(\alpha_{10}) = 0, \quad \gamma_1(\alpha_{10}) = 0.
\]

We suppose that the radius of the base of the cornea is constant for different corneas, and the value of $\alpha_{10}$ depends on the radius of curvature of the cornea $R$. 

25
In the pole three symmetry conditions take place \( u(\pi/2) = 0, \ w'(\pi/2) = 0, \ \gamma_1(\pi/2) = 0 \).

3. Numerical solutions

The problem was solved by means of the method of consequent loading (delta method) [3]. The load \( P = 10 \text{ g} \) is represented as a sum

\[
P = \Delta P_1 + \Delta P_2 + \ldots + \Delta P_n, \quad (\Delta P > 0, n \gg 1)
\]

On each step we consider a linear problem, but with new Lame’s coefficients and curvature, that describe the geometry of the deformed shell on the previous step. The increment size \( \Delta P_i \) could be different for each step. The applied flat-bottomed load creates stresses only on the contact area and it is necessary also to control and to redistribute the stress from the outer loading [3].

To solve this boundary value problem a program based on finite-difference numerical method was developed in code Mathematica 8.0. During the calculations we keep track of the outer radius of contact area \( R_{out} \) and the radius of inner unloaded area \( R_{in} \).

Results of simulations for cornea of different radius of curvature \( R \) and thickness \( h_c \) are presented in Table 1. It is supposed that intraocular pressure is \( p = 15 \text{ mm Hg} \). The following values are used for modules of elasticity and Poisson’s ratio [1]

\[
E_1 = E_2 = 0.25 \text{ MPa}, E_3 = \frac{E_1}{100}, G_{13} = 0.25 \cdot 10^5 \text{MPa},
\]

\[
\nu_{12} = 0.45, \quad \nu_{13} = \nu_{23} = 0.01.
\]

### Table 1. Radii of the contact area \( R_{out} \) and the inner unloaded area \( R_{in} \) for load \( P = 10 \text{ g} \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( R = 8 \text{ mm} )</th>
<th>( R = 9 \text{ mm} )</th>
<th>( R = 10 \text{ mm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( R_{out} )</td>
<td>( R_{in} )</td>
<td>( R_{out} )</td>
</tr>
<tr>
<td>4.2 mm</td>
<td>1.62</td>
<td>0.69</td>
<td>1.78</td>
</tr>
<tr>
<td>5.2 mm</td>
<td>1.47</td>
<td>0.50</td>
<td>1.54</td>
</tr>
<tr>
<td>6.2 mm</td>
<td>1.32</td>
<td>0.22</td>
<td>1.57</td>
</tr>
</tbody>
</table>

It’s natural that decreasing of central cornea thickness leads to increasing of the contact areas. For load \( P = 5 \text{ g} \) the area of detachment is not obtained.

Closed results obtained with the method of minimization of shell elastic potential and with finite element method.

The contact stresses for cornea with \( R = 8 \text{ mm} \) and \( h_c = 5.2 \text{ mm} \) are represented in Fig. 2.

4. Conclusions

All results show that the flatter is the form of the cornea the larger is the radius of the contact area with the applied flat-bottomed load and the larger is the radius of the inner unloaded area.

Statistical analysis show that intraocular pressure obtained by Maklakoff’s tonometer are less sensitive to the cornea thickness than Goldmann or pneumotonometers [4]. However the latest research shows [5] that for “plate” cornea Maklakoff’s tonometer gives
understated value of intraocular pressure. This can be explained by loss of contact between the cornea and the tonometer.

![Contact stresses](image)

Fig. 2. Contact stresses.

References


Abstract. Numerical analysis is performed in order to investigate deformation behavior and strength of symmetric and antisymmetric laminates. The quasi-isotropic laminate represents a good compromise in the case of in-plane loading. Antisymmetric orientation of external layers of in-plane balanced laminate can be used to ensure the necessary adaptive warping and strength of composite under action of tensile stresses. The efficiency of the use of carbon and wooden laminate (birch plywood) in wind rotor blades is analysed.

Keywords: laminates, compliance, stiffness, force stress, couple stress, curvature.

1. Introduction

In many applications the designer is concerned with the strength as well as the stiffness of composite laminates. Unlike stiffness, the strength of composite materials is not easy to predict. This is the result of the existence of many complex modes of the failure. On the other hand, laminates can be designed to provide the desired strength and stiffness characteristics required for specific applications under action of mechanical or hygrothermal loads. The material anisotropy observed in some laminate configurations can be exploited to induce coupling between deformation modes what promote to attain beneficial effects.

There are many applications where the design of a laminate is mainly governed by in-plane strength and stiffness requirements. Examples of a challenging lightweight structure made of multi-material laminates are wind rotor blades as well as aircraft wings [1]. The rotor blades are usually made using fibreglass mats, which are impregnated with polyester or epoxy. The blades may be made wholly or partially from carbon fibre, which is a lighter, but costlier material with high strength [2]. Wood-epoxy laminate is reasonably priced, domestically available, ecologically sound, and easily fabricated with low energy consumption [3].

The use of fibre reinforced composite rotor blades enables a number of possible passive aerodynamic control options. However, considering the membrane-bending stiffness coupling effects increase the complexity of the design process. In laminate design, the objective function could be strength or strain characteristic, while constraints are imposed on other properties [4]. In practice, fibre-reinforced composite laminates for conventional stiff rotor blades incorporate a combination of unidirectional plies to support axial loads and provide sufficient bending stiffness, and 45° plies to restrict shear and torsion. Representing the pertinent laminate characteristic as a function of undetermined skin layer orientation angle $\varphi$ the laminate configuration design can be solved as an optimisation problem.
The quasi-isotropic laminate represents a good compromise in the case of in-plane loading. For in-plane balanced laminates with anti-symmetric orientation of external layers can be ensured adaptive warping deformation and strength of composite laminate.

2. Stress and strain relationships of layered structure

Within a layered composite of thickness \( h \), deformation of one layer is constrained by the other ones of different orientations, and hence stresses arise in each layer. In general case, the stresses in the elementary layers are different and the stress state of the composite is inhomogeneous. By using a static equivalent system of average force stresses \( \sigma_j \) and moment stresses \( \mu_j \) acting on a composite material, the constitutive relations for the mid-plane strains \( \varepsilon_i^0 \) and the curvatures \( \kappa_i \) in matrix notations are given by [5]

\[
\begin{bmatrix}
\varepsilon_i^0 \\
\kappa_i
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
\beta^T & \delta
\end{bmatrix} \begin{bmatrix}
\sigma_j \\
\mu_j
\end{bmatrix},
\]

(1)

where \( \alpha, \beta, \delta \) are compliances of layered composite; superscript \( T \) denotes transposition operation.

The force stresses and moment stresses in layered composite are calculated by averaging

\[
\sigma_j = \int_{-h/2}^{h/2} \tilde{\sigma}_j^{(k)} \, dx_3, \quad (2)
\]

\[
\mu_j = \int_{-h/2}^{h/2} \tilde{\sigma}_j^{(k)} x_3 \, dx_3. \quad (3)
\]

The stresses \( \tilde{\sigma}_j^{(k)} \) in the \( k \)-th elementary layer (Fig. 1) in the coordinates of composite \( \{ x_i \} \) \( (i = 1, 2, 3) \) can be determined by using the layer stiffness \( A'_i j \) in the local coordinate system \( \{ x_i' \} \) and stress transformation matrix [6].

![Fig. 1. Multilayer model of composite structure.](image)

In the general case, the compliance matrices in (1) are represented in terms of composite stiffness:

\[
\alpha = S + SBCBS, \quad (4)
\]

\[
\beta = -SBC, \quad (5)
\]

\[
\delta = C \quad (6)
\]

with
The stiffness components in (4)–(7) are evaluated by integration:

\[
[A_{ij}, B_{ij}, D_{ij}] = \int_{-h/2}^{h/2} A^{(k)}_{ij} [1, x_3, x_3^2] dx_3.
\]

(8)

The compliance matrix in (4), (5) and (7) is determined by inversion procedure

\[
[S_{ij}] = [A_{ij}]^{-1}.
\]

3. Strength analysis of designed laminate

The most frequently used failure criterion is polynomial criteria advised by Tsai and Wu [7]. A more general form of the failure criterion for orthotropic materials, i.e. materials with two mutually perpendicular planes of symmetry in mechanical properties, under plane stress state is expressed as

\[
f(\sigma_{ij}) = F_{11}\sigma_{11} + F_{22}\sigma_{22} + 2F_{12}\sigma_{12} + F_{1111}\sigma_{11}^2 + F_{2222}\sigma_{22}^2 + 4F_{1212}\sigma_{12}^2 + 2F_{1222}\sigma_{11}\sigma_{22} + 4F_{1112}\sigma_{11}\sigma_{12} + 4F_{2212}\sigma_{22}\sigma_{12} = 1.
\]

(9)

The coefficients in strength function (9) are components of tensors \( F_{ij} \) and \( F_{ijkl} \). They are determined by means of the strength tensor \( R_{\alpha\beta\gamma} \), where \( \alpha = 0, 1, \overline{11} \); \( \beta = 0, 2, \overline{22} \); \( \gamma = 0, 12, \overline{12} \). The index 0 denotes that the given stress component is absent; the bar over the index is employed to indicate a compressive component. Using the strength values obtained experimentally, the coefficients are represented in the following form:

\[
\begin{align*}
F_{11} &= \frac{R_{\overline{11}00} - R_{1100}}{R_{1100}R_{\overline{11}00}}; \\
F_{22} &= \frac{R_{0220} - R_{0022}}{R_{0220}R_{0022}}; \\
F_{1111} &= \frac{1}{R_{1100}R_{\overline{11}00}}; \\
F_{2222} &= \frac{1}{R_{0220}R_{0022}}; \\
2F_{1122} &= \frac{F_{11} - F_{22}}{R_{11220}} + F_{1111} + F_{2222} - \frac{1}{R_{11220}}; \\
4F_{1212} &= \frac{1}{R_{0012}R_{0012}}.
\end{align*}
\]

(10)

The mentioned criterion defines an envelope in stress space, i.e. if the stress state lies outside of this envelope, and then failure is predicted.

4. Laminate in-plane stiffness design

Let us examine optimal design of symmetric carbon epoxy laminate [5] for two load conditions: axial force \( N_1 = 1800 \, \text{kN/m} \) and in-plane shear force \( N_{12} = 600 \, \text{kN/m} \). For every loading condition, the allowable laminate normal strain \( \varepsilon_{1,\text{allow}} \) and shear strain \( \gamma_{12,\text{allow}} \) is limited. In order to find the lightest laminate the following laminates with three layers orientations are considered: 1) \( [0]_n \), 2) \( [\pm 45]_n \), and 3) \( [(\pm 45/90)/0]_n \). For each laminate, it is necessary to determine the smallest value of \( n \) that satisfies the specified strain limits, where \( n \) is the number of layers in the case of first laminate, but it is the number of sub laminates in half of the laminates for the second and third case.

The strain limit of laminate \( [0]_n \) for the first load case may be expressed as

\[
\varepsilon_1 = \frac{N_1}{hE_1} \leq \varepsilon_{1,\text{allow}};
\]

(11)
and for the second load case

\[ \gamma_{12} = \frac{N_{12}}{hG_{12}} \leq \gamma_{12,\text{allow}}, \]  

(12)

where \( E_1 \) and \( G_{12} \) are effective elastic modulus and shear modulus of a laminate.

Considering graphite epoxy laminate with layer thickness \( h^{(k)} = 0.13 \text{ mm} \) on the basis of restrictions (1) and (2) for all zero laminate the total thickness \( h \) for first load case is \( h \geq 3.4 \text{ mm} \) but for second load case \(- h \geq 13.7 \text{ mm} \). Since the thickness \( h \) has to be an integer multiple of the layer thickness, the total number of layers is 106 and \( h = 13.8 \text{ mm} \). For the \([\pm 45]_n\) laminate the equation (11) gives \( h \geq 20.0 \text{ mm} \). The laminate has repeating stacks and is symmetric. Therefore, the total number of stacks is 80 and \( h = 20.8 \text{ mm} \). In a similar way, for the \([\pm 45/90/0]_{n_5}\) laminate the equation (11) gives \( h \geq 8.4 \text{ mm} \). In that way, the total number of stacks is 18 and the total thickness \( h = 9.4 \text{ mm} \).

Note that the all zero laminate \([0]_n\), is very flexible in shear. In the second case, the laminate \([45/-45]_{40s}\) is too flexible in the axial direction. The quasi-isotropic laminate \([\pm 45/90/0]_{9s}\) is the lightest laminate and represents a good compromise for both load cases.

5. Optimisation of layers orientation

In practice, fibre-reinforced composite laminates for conventional stiff rotor blades incorporate a combination of unidirectional plies to support radial loads and provide sufficient bending stiffness, and 45° plies to restrict shear and torsion. In the case of fixed thickness, the number of plies is not a design variable. Representing the pertinent characteristic as a function of undetermined skin layer orientation angle \( \varphi \) and determining the optimum can be solved as optimisation problem. Graphical procedure can be used for the design of laminates with prescribed in-plane properties [8]. The procedure is suitable for in-plane balanced angle-ply laminates made up of stacks of layers with different orientation angles \( \pm \varphi \). In addition to the balanced angle-ply groups, unidirectional layers with principal material axis aligned with the axes of the laminate, i.e. 0° and 90°, can be included in the stacking sequence.

For an in-plane balanced composite laminates elastic characteristics of the laminate can be determined by using lamination parameters. The lamination parameters contain the relevant information associated with the stacking sequence and they can be determined in the following way

\[ V_1(A,B,D) = \int_{-h/2}^{h/2} \cos 2\varphi \{1, x_3, x_3^2\} dx_3; \]  

(13)

\[ V_3(A,B,D) = \int_{-h/2}^{h/2} \cos 4\varphi \{1, x_3, x_3^2\} dx_3. \]  

(14)

In general case, lamination parameters and material invariants [9] can be used for determination of stiffness characteristics \( A_{ij}, B_{ij} \) and \( D_{ij} \).

By using lamination parameter diagram (Fig. 2), it is possible to determine the region of allowable combinations of lamination parameters. For a laminate of total thickness \( h \),
where the volume fraction of layers with \( \pm \varphi_i \) orientation angles is \( v_i \), normalized lamination parameters are given as

\[
\bar{V}_1 = \frac{V_{1A}}{h} = \sum_{i=1}^{N} v_i \cos 2\varphi_i ;
\]

\[
\bar{V}_3 = \frac{V_{3A}}{h} = \sum_{i=1}^{N} v_i \cos 4\varphi_i ,
\]

where \( N \) is the number of different \( \pm \varphi \) groups.

Values of all possible combinations of the lamination parameters are located along boundary line ABC (Fig. 2). The points A, B and C correspond to laminates with 0°, ±45, and 90° orientation angles, respectively. Any point inside the boundary line corresponds to laminates with two or more fiber orientation.

Fig. 2. Lamination parameter diagrams: ABC – defined stacks of layers; LMN – analysed laminates.

The analysed laminates consist of 9 layers that form in-plane balanced antisymmetric system with stacking sequence \([\varphi, -45, -45, 90, 0, 90, 45, -45, -\varphi]\). In the lamination parameter diagram (Fig. 2) point L corresponds to the orientation of external layers \( \varphi = 0^\circ \), but point N – to \( \varphi = 90^\circ \). Point M belongs to the laminate configuration with \( \varphi = \pm 25^\circ \), and laminate indicate the maximum warping under action of axial load.

In the Fig. 3 and Fig. 4 the strength function determined according to (9) and curvature of the laminate determined by using relationship (1) is shown for two types of layered composites – carbon epoxy and wooden laminate. The warping of wooden laminate is more in comparison with carbon epoxy laminate, but in both cases there are allowable intervals, where curvature changes from zero until maximum, but stress level is allowable.

6. Conclusions

The quasi-isotropic laminate represents a good compromise in the case of in-plane loading. The anti-symmetric orientation of skin layers of in-plane balanced laminate can be used to ensure the necessary adaptive warping and strength of the laminate under action of axial
force. The use of stretching-twisting coupling can be applied in the rotor blade to provide a control mechanism which does not have any parts moving relative to each other, and which is therefore maintenance-free.

Fig. 3. Strength function (1) and curvature (2) vs. external layer orientation for carbon epoxy laminate.

Fig. 4. Strength function (1) and curvature (2) vs. external layer orientation for wooden laminate.

References


Crack localization in Euler-Bernoulli beams

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Abstract. The crack localization problem frequently occurs in the industrial constructions. There are several analytical and numerical solutions to the problem in literature; however they do not always suit to the inverse problems or are not applicable to large structures and require complicated time-consuming calculations. The aim of the present paper is to describe a contemporary time-efficient method for crack localization in vibrating Euler-Bernoulli beams using mode shapes, Haar wavelets and artificial neural networks. Once the system is built and trained, it is not to be reprogrammed. Furthermore, the proposed method makes reasonable parameter predictions in comparison to the conventional approaches based only on natural frequencies and neural networks.

Keywords: cracks, Euler-Bernoulli beam, artificial neural networks, Haar wavelets.

1. Introduction

Cracks in supporting constructions and machinery appear mainly as a result of manufacturing defect or severe exploitation. They present an imminent threat to the whole structure since cracks change reliable behaviour of the construction. Therefore, the discovery and localization of the cracks are essential issues in the health monitoring of the structure.

It is widely accepted that one of the most efficient ways to discover damages (including cracks) is the frequency analysis. According to Konar and Chattopadhyay, vibrations are still the best [1] and one of the easily available [2] indicators of the structure's overall technical condition since each mechanical fault generates vibrations in its own specific frequency domain [3].

In the case of inverse problems, it is a challenging task to localize cracks in an Euler-Bernoulli beam using only the frequency domain. In order to obtain trustworthy results, both a proper analytical model and an efficient numerical technique have to be combined. Generally, cracks are modeled as massless rotating springs; their equivalent stiffnesses are computed as functions of crack depths using the fracture mechanics methods [4]. The analytical methods of crack detection can be based on transverse vibration, longitudinal vibration or a coupling of transverse, longitudinal and torsional vibration; however, the transverse vibration is more applicable for inspection of high volume and low cost components feasibility [5]. Alternatively, a cracked beam can be described using numerical analysis such as finite element method and the p-version of the method [6], Galerkin method or Rayleigh-Ritz method. Maghsoodi, Ghadami et al. proposed a more computationally efficient method which is based on the flexibility damage index [7]. A contemporary approach of natural frequencies and artificial neural networks (NN) was presented by Rosales, Filipich et. al in [2]. The disadvantages of the methods are either mathematical complexity or low accuracy. The aim of the present paper is to describe a new approach for crack localization taking into consideration the disadvantages of the previous methods. The novelty of the technique
consists in the combination of the first mode shape, Haar wavelets and NN-s. The significance of the solution lies in its ability to make fast accurate model-independent predictions calculating only one natural frequency and training the network only once.

2. Problem formulation

In the present research, it is assumed that an open crack with depth $a$ locates at distance $x_c$ from the left end of the Euler-Bernoulli beam; the rotational crack compliance is dominant and the bending spring constant $K_b$ in the vicinity of crack is given by [8]:

$$K_b = \frac{1}{c}, \quad c = \frac{5.346 h}{EI} J \left( \frac{a}{h} \right),$$  \hspace{1cm} (1)

where $h$ is the depth of the beam, $E$ is the Young modulus, $I$ is the moment of inertia and $J(a/h)$ the dimensionless local compliance function. In the present paper, the function proposed by Paipetis and Dimarogonas [8] is used

$$J \left( \frac{a}{h} \right) = 1.8624 \left( \frac{a}{h} \right)^2 - 3.95 \left( \frac{a}{h} \right)^3 + 16.375 \left( \frac{a}{h} \right)^4 - 37.226 \left( \frac{a}{h} \right)^5 +$$
$$+ 76.81 \left( \frac{a}{h} \right)^6 - 126.9 \left( \frac{a}{h} \right)^7 + 172 \left( \frac{a}{h} \right)^8 - 143.97 \left( \frac{a}{h} \right)^9 + 66.56 \left( \frac{a}{h} \right)^10.$$ \hspace{1cm} (2)

Due to localized crack effect, the cracked beam can be simulated as two uniform beams joined together by a spring at the crack location. The boundary conditions for the cantilever at beam ends could be expressed as follows

$$W_1(0) = \frac{dW_1}{dx}(0) = 0,$$
$$\frac{d^2W_2}{dx^2}(L) = \frac{d^3W_2}{dx^3}(L) = 0,$$ \hspace{1cm} (3)

where $L$ is the length of the beam; $W_1$ and $W_2$ denote the mode shapes on the left and right beam sections, respectively. At the crack location $x_c$ the continuity conditions have the form

$$W_1 = W_2,$$
$$\frac{d^2W_1}{dx^2} = \frac{d^2W_2}{dx^2},$$
$$\frac{d^3W_1}{dx^3} = \frac{d^3W_2}{dx^3},$$
$$\frac{dW_1}{dx} + \frac{1}{K_b} \frac{d^2W_1}{dx^2} = \frac{dW_2}{dx}.$$ \hspace{1cm} (4)

3. Complex approach

The Haar wavelets are one the simplest discrete orthogonal wavelets which are discontinuous and resemble a step function. They have been chosen in the present paper due to their ability to perform discrete wavelet transform and efficient preprocessing for damage monitoring.

The neural networks (NN) have been known as a powerful tool in the field of forecast due to their special ability to simulate complicated systems and find relationship between the
input and output data independently of the governing equations. In the numerical examples, the supervised multi-layer feedforward back propagation NN has been chosen since it is one of the most popular architectures used today [9].

The complex approach for crack localization consists of the following steps:
- calculate the first mode shape;
- transfer the mode shape into Haar coefficients;
- construct and train the NN.

Once the steps are completed, the system is ready for use and does not have to be reprogrammed.

In Table 1 are shown the predictions of the crack location and depth using six frequencies and NN (columns three and four) and only one frequency, Haar wavelet and the same NN (column five and six). The accuracy of the predictions were estimated by chi-square test: the smaller $\chi^2$, the more accurate are the predictions.

Table 1. Prediction of the location and the size of the crack in cantilever.

<table>
<thead>
<tr>
<th>Exact Location</th>
<th>Prediction based on six frequencies</th>
<th>Prediction based on mode shape and Haar transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location</td>
<td>Depth</td>
<td>Location</td>
</tr>
<tr>
<td>0.18</td>
<td>0.21</td>
<td>0.2148</td>
</tr>
<tr>
<td>0.26</td>
<td>0.37</td>
<td>0.3082</td>
</tr>
<tr>
<td>0.34</td>
<td>0.37</td>
<td>0.3498</td>
</tr>
<tr>
<td>0.5</td>
<td>0.49</td>
<td>0.5010</td>
</tr>
<tr>
<td>0.58</td>
<td>0.25</td>
<td>0.6124</td>
</tr>
<tr>
<td>0.66</td>
<td>0.25</td>
<td>0.6721</td>
</tr>
<tr>
<td>0.74</td>
<td>0.17</td>
<td>0.7674</td>
</tr>
<tr>
<td>0.82</td>
<td>0.33</td>
<td>0.8580</td>
</tr>
<tr>
<td>0.9</td>
<td>0.25</td>
<td>0.9000</td>
</tr>
<tr>
<td>$\chi^2$</td>
<td>2.08E-02</td>
<td>5.25E-03</td>
</tr>
<tr>
<td>Total $\chi^2$</td>
<td>2.60192E-02</td>
<td>2.76E-03</td>
</tr>
</tbody>
</table>

4. Summary

A complex approach for crack identification in Euler-Bernoulli beams has been described in the present paper. The results were compared to the results obtained using a conventional method of frequencies and NN. The contemporary method of Haar wavelets has shown reasonable results which could be applicable for structure health monitoring systems.
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References


Dynamic grounding analysis

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Abstract. The paper presents a method to assess oil tanker grounding damage through dynamic simulation. This means that the actual ship motions during the grounding are directly included in the assessment of structural damage. Therefore, the hydrostatic restoring forces, hydrodynamic non-structural mass (added mass) and actual structural mass are included in the FE analysis. Since the analysis of entire tanker would be too time consuming, only a part, which interacts with the rock, is modelled. The remaining ship mass and inertia is modelled using finite number of mass points. The hydrostatic restoring forces are also modelled through mass points. As the mass point displaces from its equilibrium position, the restoring force proportional to this displacement is applied. The methodology is applied to simulate a grounding of a chemical tanker.

Keywords: dynamic grounding simulations.

1. Introduction

Numerical simulations are often used to assess the grounding damage of the ship. Such analysis is a demanding task and usually significant simplifications are made. The analysis can be split into two: (1) inner mechanics (2) outer dynamics. The inner mechanics deals with the investigation of the structural behaviour of the grounding ship and evaluates the contact force and energy as a function of penetration depth. The external dynamics looks into the motions of the ship and evaluates the energy involved with ship motions. In typical grounding analysis these two fields are treated separately and the simulation is displacement controlled: the ship is fixed and rock moves with constant speed along the ship. This is called a decoupled simulation and less precise damage description is obtained as actual ship motions are not evaluated. Decoupled analysis are presented among others for example in Törnvqvist [6], Alsos and Amdahl [1], Simonsen et al [5] So-called coupled simulations provide more precise outcome as the inner mechanics and outer dynamics are evaluated simultaneously. Coupling of the inner mechanics and the external dynamics means that both are included in one simulation model and evaluated during the same calculation run. Dynamic analysis are discussed, for example, in Nguyen et al [2], Pill and Tabri [3].

This paper proposes a numerical coupled approach, where ship mass and inertia is included in the analysis to consider precise ship motions. The mass of a fully loaded ship is sum of ship structural mass and additional mass (cargo, machinery, etc.). Part of the structural mass is already included in the FE model, but additional mass is modelled using a finite number of mass points. The correct rotational inertia is assured by the position of mass points. During the grounding the ship experiences heave, roll and pitch motions causing sections to immerge or emerge. In order to restore the position of the ship corresponding restoring forces and moments are initiated. The corresponding hydrostatic restoring forces depend mainly on displacement, therefore, to describe these forces with acceptable accuracy, the ship is divided into virtual sections in longitudinal and transverse directions.
he buoyancy points are defined for these sections and restoring force acting to these points is described as a function of z coordinate. This method allows to obtain the restoring forces by summing forces acting on individual sections. As a result, such dynamic analysis gives a description of structural damage that is in correlation with the kinetic energy of a grounding ship and rock topology.

2. Methodology for dynamic grounding simulations

Dynamic grounding simulations imply the simulations where inner mechanics and external dynamics are treated simultaneously during the same calculation run. This requires the inclusion of all the major forces acting at the grounding ship. Hydromechanical forces and moments acting on a surface ship consist of water resistance, hydrostatic restoring forces, and radiation forces expressed in terms of hydrodynamic damping and added mass. Grounding introduces contact force and inertial forces. To consider all the forces in numerical simulations, ship mass, inertia and hydromechanics parameters are to be included in the simulation model. This section describes the procedures for dynamical grounding simulation of a fully loaded ship.

2.1. Ship mass and inertia

The true motions of the ship are obtained through the correct modelling of ship inertias. Since the fully loaded ship is to be modelled in this paper then the mass of a fully loaded ship $\Delta$ is sum of ship structural mass and cargo mass. Part of the structural mass is already included in the FE model, denoted hereinafter as $m_{str}$. Rest of the mass, denoted as $m_{mp}$, is modelled using a finite number $N_m$ of points with equal mass. The total mass of the ship can then be presented as

$$\Delta = m_{str} + m_{mp},$$

where

$$m_{mp} = \sum_{i=1}^{N_m} m_i.$$  \hspace{1cm} (2)

The total rotational inertia $I^j_\Delta$ of a mass $\Delta$ is a sum of inertias given as

$$I^j_\Delta = I^j_{str} + I^j_{mp},$$  \hspace{1cm} (3)

where $I^j_{str}$ is part of the inertia which is already included in the model through mass $m_{str}$, the inertia $I^j_{mp}$ is to be described by proper positioning of mass points. Hereinafter, the superscript $j$ denotes the axis of rotation. Correct modelling of rotational inertia is assured by the position of mass points. Expressing the position with average distance, see Fig. 1, gives an inertia $I^j_{mp}$ that can be presented in simple form as

$$I^j_{mp} = 4 \cdot s^2 \cdot \frac{m_{mp}}{N_m}.$$  \hspace{1cm} (4)

Substituting equation (4) into (3) gives the average distances $s_x$ and $s_y$ of mass points as follows
\[ s_x = \frac{(I_y^\Delta - I_y^{str}) \cdot N_m}{4 \cdot m_{mp}}; \quad s_y = \frac{(I_x^\Delta - I_x^{str}) \cdot N_m}{4 \cdot m_{mp}} \] (5)

The nodal points are defined for the definition of mass points (Fig. 1). To attach mass points to the ship, rigid parts are first created for the aft and fore end of the ship. For the definition of rigid parts 50 cm of cross-section in longitudinal direction at both ends of the ship are modelled as rigid and the mass points are constrained to move together with these rigid bodies.

2.2. Hydrodynamic added mass

As the velocity of the ship changes rapidly due to the grounding, the hydrodynamic added mass and damping starts to play important role. The added mass is included as a nodal mass at the centre of gravity (COG) of the ship. As the added mass is different for all the motion components, the complete mass matrix is described for the mass point through *ELEMENT_MASS_MATRIX keyword in LS-DYNA. Only the diagonal elements of the mass matrix are considered, the influence of other components is considered neglectful.

\[ \mathbf{M}_{roll} = \mathbf{V} \cdot \rho \cdot g \cdot G M_T \cdot \theta \] (6)

where, \( \mathbf{V} \) is the displacement of the ship, \( G M_T \) transversal metacentric height, \( \theta \) roll angle, \( \rho \) density of the seawater and \( g \) gravitational acceleration.

It is known from the ship theory that the position of COB changes with respect to the COG. Since it is difficult to model locally moving COB the simplification for this methodology is made and COB is considered as fixed with respect to the ship. The

Fig. 1. The principle scheme of distribution of mass and buoyancy points, a) top view, b) front view.

2.3. Restoring force

When the ship floats the buoyancy force acts through the centre of displaced volume, i.e. centre of buoyancy (COB) balancing the ship weight which acts through the ship’s COG. During the ship grounding, translational (heave) and rotational motions (roll, pitch) are induced due to the ship interaction with the rock. Thus the restoring forces and moments are induced. According to the ship theory the roll restoring moment of the ship in case of small roll angles can be obtained with (Rawson and Tupper [4])

\[ M_{roll} = \mathbf{V} \cdot \rho \cdot g \cdot G M_T \cdot \theta \] (6)

It is known from the ship theory that the position of COB changes with respect to the COG. Since it is difficult to model locally moving COB the simplification for this methodology is made and COB is considered as fixed with respect to the ship. The
buoyancy points are defined and attached in FE model with the same procedures as mass points. To model the restoring moments in FE model the ship is divided longitudinally and transversally into \( N_b \) equal sections and the restoring force is applied to buoyancy point corresponding to that section, see Fig. 1. In this methodology the roll restoring moment is modelled with buoyancy points as

\[
M_{roll} = \rho \cdot g \cdot \sum_{i=1}^{N_b} A_i \cdot T_i \cdot y_i \tag{7}
\]

were, \( A_i \) is section area and \( T_i \) draught of the \( i \)-th immersed section, \( y_i \) is the distance between section buoyancy point and \( x \)-axis. The pitch restoring moment is given by following the same principle with the difference that \( BM_L \) is used as metacentric radius and \( x_i \) is used instead \( y_i \). When ship floats in the sea at design draught \( T \), the buoyancy force applied to each point is

\[
F_{buoy} = \frac{\rho \cdot g \cdot A \cdot T}{N_b} \tag{8}
\]

where, \( A \) is ship’s section area at water level. During the grounding the ship experiences heave, roll and pitch motions causing sections to immerse or emerge and therefore increasing or decreasing the restoring force. This change is described with \( z \)-coordinate of buoyancy points. Since the ship model is located so that the centre of gravity and origin of the global coordinate system coincide then the buoyancy force given with (8) applies when \( z = z_{bot} - T/2 \). The restoring force is applied to the buoyancy points with the load curve described as a linear relation between the force and displacement. The curve ensures that no force is applied at coordinate value \( z = (z_{bot} - T/2) - T \).

3. Case study: grounding of a chemical tanker

The dynamic grounding simulation was conducted to test the methodology described in section 2. The simulation is arranged so that the tanker with given initial velocity of 15 knots (~7.8 m/s) runs over a rigidly modelled rock which is fully fixed. The rock is paraboloidal axisymmetric which height is 3 m and width 8.5 m. The penetration depth is chosen such that the penetration depth exceeds the tank top height by 0.5 m. The simulation lasts until the ship speed decreases to zero.

3.1. Description of a tanker

The particulars of the oil tanker used in case study are shown in Table 1. The fully loaded condition is used. FE model of the ship is shown on Fig. 2.

<table>
<thead>
<tr>
<th>Table 1. Main dimensions.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Length over all</strong></td>
</tr>
<tr>
<td><strong>Length between perpendiculars</strong></td>
</tr>
<tr>
<td><strong>Breadth, moulded</strong></td>
</tr>
<tr>
<td><strong>Depth to upper deck</strong></td>
</tr>
<tr>
<td><strong>Draught design</strong></td>
</tr>
<tr>
<td><strong>Displacement in salt water at ( T = 11.10 \text{ m} )</strong></td>
</tr>
</tbody>
</table>
3.2. Results

The penetration induced during dynamic grounding is shown on Fig. 2. Simulation results clearly show the importance of ship motions when evaluating the bottom damage in ship grounding. Simulation showed that while the ship experiences pitch and roll motions the vertical penetration depth changes. As the rock moves closer to centre of the ship the penetration depth stabilises. The velocity of the ship decreases almost linearly throughout the grounding.

![Fig. 2. Grounding damage.](image)

The Fig. 3 presents the restoring force of a selected point and contact force acting between the ship and the rock. The Fig. 3a) shows that the restoring force decreases for the emerging buoyancy points and increases for the immersing points. This indicates that restoring forces are correctly applied.

![Fig. 3. a) restoring force; b) contact force.](image)

Conclusions

The method to assess oil tanker grounding damage through dynamic simulation was presented in this paper. Hydrostatic restoring forces, hydrodynamic added mass and actual structural mass were included in the FE analysis. The full mass of the ship was modelled with the aid of defined mass points. The restoring force was modelled with the buoyancy points. Although in principle the method works, the current application of inertial and buoyancy forces results in high force and stress concentrations at the ends of the ship, where rigid bodies transfer the inertial and buoyancy forces to the ship hull. This results in unrealistically high loads at the ship hull that induces buckling of the deck stiffeners. Therefore the methodology still needs further improvement. Inertial forces and the restoring force should be divided more evenly over the whole ship to prevent force concentrations closed to the rigid bodies and the end of the ship. Inertial forces can be
included by increasing the material density and the restoring forces can be modelled as a distributed pressure over the whole ship.

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References

A theory of coupled beams for non-prismatic ship structure

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Abstract. This paper presents the method that enables to assess the bending strength of a multi-deck ship. The method is applied for simple plate structures including non-prismatic cross-section and openings and validated with the three-dimensional finite element method. The results show a good agreement between these two methods.

Keywords: coupled beam method, finite element method, ship hull girder bending, hull superstructure interaction.

1. Introduction

At present a practical tool to estimate the global bending response of a modern passenger ship is the three-dimensional (3D) finite element (FE) method. However, the drawback of the FE method is that it is time-consuming. Therefore, simplified methods are useful in the preliminary design stage, as they enable the calculation of numerous different cases over a relatively short period of time in order to find the best solution. Crawford [1] was the first to develop the method based on the two-beam theory, taking into account the longitudinal shear force and the vertical force due to the hull-superstructure interaction. Bleich’s approach [2] is similar, describing a straightforward computation of normal stresses for prismatic beams. The coupled beam (CB) method is based on the beam theory given by Bleich and extended to multi-deck superstructures. The applicability of the CB-method for prismatic structures has been validated by Naar, Varsta and Kujala [3].

In this paper, the coupled beam CB-method is used to estimate normal and shear stresses and vertical deflection of three different types of plate structures. The applicability of the method in case of non-prismatic plate structures and plate structures with openings is verified with the three-dimensional finite element method.

2. Description of the method

The hull girder is divided into longitudinal beams that have bending and axial stiffness. Each beam consists of one deck structure with a connected side shell or side shells. The cross-section may need to be divided into beams not only in the vertical direction but for some decks also in the horizontal direction as shown in Fig. 1. The beams are connected by distributed springs, which transfer vertical forces and longitudinal shear forces between the beams. The stiffness of springs corresponds to the vertical elongation of the bulkhead or the side shell and to the shear deformation of the structure connecting two decks. A detailed description of the CB-method is given by Naar, Varsta and Kujala [3].
In the paper presented by Naar et al. [3] the axial displacement and deflection variables are approximated as the linear combination of the known shape functions $B(\xi)_1, \ldots, B(\xi)_m$ and unknown constants $c_1, \ldots, c_m$, where $\xi$ is a non-dimensional coordinate. Each function $B(\xi)_i$ is valid over the entire length of the ship. In the present analysis, the structure is longitudinally divided into shorter line segments. In each segment displacement $u$ and deflections $v^Q$ and $v^M$ are described by the shape functions valid only in this particular segment. The shape functions for displacement $u$ and deflection $v^Q$ caused by the shear deformation are

$$B(\xi)_1 = 1 - \xi$$

and

$$B(\xi)_2 = \xi$$

and the shape functions for deflection $v^M$ caused by the bending deformation are

$$B(\xi)_1 = 2\xi^3 - 3\xi^2 + 1,$$

$$B(\xi)_2 = h_e(\xi^3 - 2\xi^2 + \xi),$$

$$B(\xi)_3 = -2\xi^3 + 3\xi^2,$$

and

$$B(\xi)_2 = h_e(\xi^3 - \xi^2),$$

where $h_e$ is the length of a line segment in the longitudinal direction.

3. Finite element and coupled beam models

In order to validate the coupled beam approach of non-prismatic ship structure three simplified shell models are analysed. The dimensions and the cross-sections of all the structures are shown in Fig. 2. The material properties of steel are used. The plate structure is attached to a 10 mm thick foundation plate with the elastic modulus 40,000 times lower than that of the steel. The dimensions of the foundation plate are $2 \times 60$ m. Thus, the foundation plate supports the lower plate in its entire length. In the coupled beam (CB) method the structures are divided into three longitudinal beams as shown in Fig. 2. First, a 60 m long and 9 m high prismatic plate structure is analysed. The finite element (FE) model includes 66,000 four node plate elements with the dimensions $10 \times 10$ mm. In the CB-method, the position of the reference line for each beam is fixed to the centroid of each cross-section, see Fig. 2. In the second case, a 60 m long non-prismatic plate structure
is analysed. The FE model includes about 47,400 four node plate elements. In the CB-method, the position of the reference line for each beam is fixed to the centroid of each cross-section, see Fig. 2. In the third case, a prismatic plate structure with openings is analysed. Apart from the openings located asymmetrically with respect to the main frame, the plate structure is the same as the prismatic one. The FE model includes about 60,880 four node plate elements. In the CB-method, the position of the reference line for each beam is chosen as shown in Fig. 2.

The loading is the same for all three cases of the different plate structures. The shape of the loading is shown in Fig. 3. The loading was calculated from equation

\[ p(x) = 60 \cdot \cos \left( \frac{2\pi x}{L} \right) \frac{N}{\text{mm}}, \]

where \( L \) is the length of the structure.
4. Validation with the Finite Element Method

4.1. Vertical deflection

As a good indicator of the global behaviour of the plate structure the vertical deflection obtained by the CB-method is compared to the results given by the FE method. Table 1 shows that the results correspond well. The inaccuracy is lower than 8 %.

<table>
<thead>
<tr>
<th>Difference [%]</th>
<th>Prismatic</th>
<th>Non-prismatic</th>
<th>With openings</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.3</td>
<td>7.2</td>
<td>1.8</td>
</tr>
</tbody>
</table>

4.2. Normal stress

The vertical distribution of the normal stress obtained by the different methods is compared in two different locations in each plate structure. Fig. 4 illustrates that the stress values obtained by the CB-method correspond well with those given by the FE method. Different cross-sections have been selected for the comparison between the coupled beams (CB) and the finite element (FE) method. The longitudinal distribution of normal stress also shows a good agreement between the CB-method and the FE method as illustrated in Fig. 5. In case of the prismatic plate structure with openings, local inaccuracies appear in the zones where the openings are. However, the average stress values obtained by the FE method are compatible with those calculated with the CB-method. The comparison between the CB-method and the FE method (FEM) has been carried out for three regions for each plate structure.
4.3. Shear stress

The shear stress has been calculated between adjacent beams, see Fig. 6.
The shear stress has been calculated between beams 1 and 2 (location 1) and between beams 2 and 3 (location 2).

The comparison between the results given by the CB-method and the FE method indicates that the longitudinal distribution of shear stress can be well estimated using the CB-method. Fig. 6 shows a good agreement between these two methods. In the region where there are openings in the shear area, the stresses calculated with the CB-method are the maximum values of each column between the openings, while the values obtained by the FE method illustrate the whole range of the stress values in each column. So, the fact that the peaks of the separate stress lines (i.e. obtained by the FE method) for each column between the openings are compatible with the stress values given by the CB-method indicates that the CB-method can be used when calculating the shear stress.

5. Conclusions

In this paper the results obtained by the coupled beams method are validated by the FE method for three different plate structures.

In general, the vertical deflection given by the CB-method corresponds well with the results obtained by the FE method in all three analysed cases of different plate structures, which indicates that the CB-method can be used to describe the global behaviour of a non-prismatic plate structure as well as a plate structure with (or without) openings.

The longitudinal distribution of both shear stress and normal stress can also be estimated using the CB-method. There are only local inaccuracies compared to the FE method in the areas near the openings. Nevertheless, the maximum values in certain areas can be calculated if necessary.

On the whole, the vertical distribution of normal stress shows that the results given by those two methods correspond best near the reference axis of each beam while the difference increases proportionally with the distance from the reference axis. As a result, dividing the structure into a larger number of lower beams leads to more accurate results.
Therefore, the influence on the results has to be considered already before dividing the plate structure into the beams and choosing the location for the reference axes. In conclusion, the CB-method is applicable for the calculation of stresses and deflections of both non-prismatic and prismatic plate structures with openings. The method gives fast and reliable estimation of stresses and deflections.

References


Buckling of beams and columns on elastic foundations

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Abstract. Beams and columns subjected to the axial pressure are studied. The beams under consideration have constant rectangular cross section and rest on an elastic foundation. The beams are weakened by cracks. Critical buckling loads are established for beams clamped at one end and free at the other end.

Keywords: beam, stability, crack, elasticity, buckling.

1. Introduction

It is somewhat paradoxical that cracks and other defects seem to be unavoidable during the manufacturing and exploring of structures. This involves the need to study the influence of cracks and crack-like defects to stability, strength and stiffness of structural elements.

As regards the stability of beams and columns an effective method for determination of critical loads is the distributed line spring method (see Dimarogonas [3]). This approach was used by Caddemi and Calio [2], Lellep and Kraav [6], also Li [7]. Zheng and Fan [9] studied hollow columns injured with cracks, Zhou and Huang [10] investigated the case of eccentrically loaded columns.

In the present paper critical buckling loads are determined for stepped beams with cracks resting on an elastic foundation.

2. Problem formulation

Let us consider a cantilever beam of constant thickness supported elastically along its length. The beam is fixed at one end and free at other end (Fig. 1). It has rectangular cross section with dimensions $b$ (width), $h$ (height) and $l$ (length) that are constant values. We assume the beams or columns to be subjected to the axial pressure loading $P$. Let the origin of the coordinates be located at the center of the bottom of the beam.

Also, let us assume the beams are weakened by cracks with the length $c_j$ as shown in Fig. 1. The aim of the paper is to determine critical buckling loads for beams resting on an elastic foundation.

Fig. 1. A beam on an elastic foundation.
3. Governing equations

Let us assume that the material of the beam is purely elastic obeying the Hooke’s law. Generalized Hooke’s law can be written as (Simitses [4], Farshad [2], Iyengar [3])

\[ M = EI_j \kappa. \]  

Here \( M = M(x) \) is the bending moment, \( \kappa \) is the strain component

\[ \kappa = -v'' \]

and \( E \) is the Young modulus.

As we study beams with rectangular cross section the moment of inertia can be written as follows:

\[ I_j = \frac{h^3b}{12}. \]

Generalized Hooke’s law can be written as (Simitses [4], Farshad [2], Iyengar [3])

\[ M'' + P v'' + \beta v = 0. \]

Here prims denote the differentiation with respect to the coordinate and \( x \), \( v \) stands for the displacement in the direction of the axis \( 0y \) and \( \beta \) is the modulus of the foundation. Taking formula (2) into account one can write (4) as

\[ (EI_j v''')'' + P v'' + \beta v = 0. \]

Now let us present the formula (5) converted into the form as follows:

\[ v'''' + \frac{\lambda_j^2}{l^2} v'' + \left( \frac{\pi}{l} \right)^4 \beta v = 0, \]

where

\[ \beta_j = \frac{12 \beta l^4}{\pi^4 Ebh_j^3} \]

and

\[ \lambda_j^2 = \frac{12 Pl^2}{Eb h_j^3}. \]

One can solve the fourth order linear equation (6) with constant coefficients as follows. We present the characteristic equation as

\[ r^4 + \frac{\lambda_j^2}{l^2} r^2 + \beta = 0. \]

It can be seen that the solution for \( r \) is given by

\[ r^2 = -\frac{\lambda_j^2}{2 l^2} \pm \frac{1}{2} \sqrt{\frac{\lambda_j^4}{l^4} - 4 \beta_j} = \sqrt{\beta_j \left( -\frac{\lambda_j^2}{2 l^2 \sqrt{\beta_j}} \pm \sqrt{\left( \frac{\lambda_j^2}{2 l^2 \sqrt{\beta_j}} \right)^2 - 1} \right)}. \]

Thus, it follows from the equation (9) that the four roots of the characteristic equations are

\[ r_{1j} = i \beta_j^{\frac{1}{4}} \left( \varphi_j - \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \quad r_{2j} = -i \beta_j^{\frac{1}{4}} \left( \varphi_j - \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \]

\[ r_{3j} = i \beta_j^{\frac{1}{4}} \left( \varphi_j + \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \quad r_{4j} = -i \beta_j^{\frac{1}{4}} \left( \varphi_j + \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \]

where \( \varphi_j = \lambda_j^2 / (2 l^2 \sqrt{\beta_j}) \).
\[ r_{3j} = i \beta_j^4 \left( \varphi_j + \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \quad r_{4j} = -i \beta_j^4 \left( \varphi_j + \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \]

where \( i \) is the imaginary unit and

\[ \varphi_j = \frac{\lambda_j^2}{2l^2 \beta_j}. \]  

(12)

It can be easily seen from the solutions (11) that there are three cases that we should consider: \( \varphi_j > 1, \varphi_j = 1, \varphi_j < 1 \).

Firstly, let us see the case of \( \varphi_j > 1 \). If we denote the real positive parameters \( s_1 \) and \( s_2 \) as

\[ s_{1j} = \beta_j^4 \left( \varphi_j - \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \quad s_{2j} = \beta_j^4 \left( \varphi_j + \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, \]  

(13)

we can write the four roots of characteristic equation (9) as follows:

\[ r_{1j} = is_{1j}, r_{2j} = -is_{1j}, r_{3j} = is_{2j}, r_{4j} = -is_{2j}. \]  

(14)

Now the real part of the general solution of the equation (6) is

\[ v_{1j} = A_{1j} \cos s_{1j} \xi + B_{1j} \sin s_{1j} \xi + C_{1j} \cos s_{2j} \xi + D_{1j} \sin s_{2j} \xi, \]  

(15)

where \( \xi = x/l \). In (15) \( A_{1j}, B_{1j}, C_{1j}, D_{1j}, j = 0, \ldots, n \) stand for arbitrary constants to be defined from the boundary and intermediate conditions later.

As boundary conditions depend on the type of support conditions, then in the case of a beam clamped at the left end and free at the right end one has at \( x = 0 \)

\[ v(0) = 0, v'(0) = 0 \]  

(16)

and at \( x = l \)

\[ v''(l) = 0, v'''(l) + \frac{\lambda_j^2}{l^2} v'(l) = 0. \]  

(17)

For the case of \( \varphi_j = 1 \) let us denote the real positive parameter \( s_{3j} \) as

\[ s_{3j} = \beta_j^4 \]  

(18)

and write the four roots of characteristic equation (9) as follows:

\[ r_{1j} = is_{3j}, r_{2j} = -is_{3j}, r_{3j} = is_{3j}, r_{4j} = -is_{3j}. \]  

(19)

Here the real part of the general solution of the equation (6) is

\[ v_{2j} = A_{2j} \cos s_{3j} \xi + B_{2j} \sin s_{3j} \xi + C_{2j} \cos s_{3j} \xi + D_{2j} \xi \sin s_{3j} \xi. \]  

(20)

And finally, if we consider the case of \( \varphi_j < 1 \), we write the four roots of characteristic equation (9) as

\[ r_{1j} = i \beta_j^4 \left( \varphi_j - i \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, r_{2j} = -r_{1j}, \]  

(21)

\[ r_{3j} = i \beta_j^4 \left( \varphi_j + i \sqrt{\varphi_j^2 - 1} \right)^{\frac{1}{2}}, r_{4j} = -r_{3j}, \]

or

\[ r_{1j} = t_j + i u_j = \eta_j, r_{2j} = -ir_{1j}, \]  

(22)
we can present the system as
\[ r_{3j} = -t_j + iu_j = \omega_j, r_{3j} = -ir_{3j}, \]
where
\[ t_j = \beta_j^4 \sqrt{\frac{1 - \varphi_j}{2}}, u_j = \beta_j^4 \sqrt{\frac{1 + \varphi_j}{2}}. \]  
(23)

Here the real part of the general solution of the equation (6) is
\[ v_{3j} = A_{3j} \cosh \eta_j \xi + B_{3j} \cosh \omega_j \xi + c_{3j} \xi \sinh \eta_j \xi + D_{3j} \xi \sinh \omega_j \xi. \]  
(24)

4. Critical buckling loads

As we show, the displacement \( v(x) \) is defined by (15), (20) or (24) depending on the value of \( \varphi \). So let us see separately the three different cases again. Firstly, let us consider the beam with constant cross section that has no cracks.

If \( \varphi > 1 \), the system of boundary conditions (16) and (17) takes the form
\[
A_1(-s_1^2 \cos s_1 + s_2^2 \cos s_2) + B_1(-s_1^2 \sin s_1 + s_1 s_2 \sin s_2) = 0,
\]
\[
A_1[s_1(s_1^2 - \lambda) \sin s_1 - s_2(s_2^2 - \lambda) \sin s_2] +
B_1[-s_1(s_1^2 - \lambda) \cos s_1 - s_1 \lambda \cos s_2 + s_1 s_2 \sin s_2] = 0.
\]  
(25)

Calculating the determinant of the system and equalizing it to the zero gives us the result in the following form:
\[
s_1^5 - s_1 \lambda(s_1^2 + s_2^2) + s_1 s_2 \sin^2 s_2 + [s_1 \lambda(s_1^2 + s_2^2) - s_1^3 s_2^2] \cos s_1 \cos s_2 +
+[-s_1^2 s_2(s_1^2 + s_2^2) + 2s_1^2 s_2 \lambda] \sin s_1 \sin s_2 - s_1^3 s_2 \cos s_1 \sin s_2 = 0.
\]  
(26)

In case of \( \varphi = 1 \) the system of boundary conditions is
\[
B_2(s_3^3 \cos s_3 + s_2^3 \sin s_3) + D_2(2s_3 \cos s_3 - s_2^2 \sin s_3) = 0,
\]
\[
B_2[s_2^3(s_3 - \lambda^2) \sin s_3 - 2s_3^2 \cos s_3] +
D_2[(s_3^2 - \lambda^2) \sin s_3 + 2s_3^2 \sin s_3 + s_3(s_2^2 - \lambda^2) \cos s_3] = 0.
\]  
(27)

The determinant of the system equalized it to the zero is:
\[
3s_3^4 + s_2^4(s_3^2 - \lambda^2) + s_2^4 \cos^2 s_3 + s_3^2 \lambda^2 \sin^2 s_3 = 0.
\]  
(28)

In case of \( \varphi < 1 \) we can present the system as
\[
A_3(\eta^2 \cosh \eta - \omega^2 \cosh \omega) + C_3(\eta^2 \sinh \eta - \eta \omega \sinh \omega) = 0,
\]
\[
A_3[\eta(\eta^2 + \lambda^2) \sinh \eta - \omega(\omega^2 + \lambda^2) \sinh \omega] +
C_3[\eta(\eta^2 + \lambda^2) \cosh \eta - \omega(\omega^2 + \lambda^2) \cosh \omega] = 0
\]  
(29)

and the determinant equalized to the zero is
\[
\eta^3(\eta^2 + \lambda^2) + \eta \omega^2(\omega^2 + \lambda^2) - \cosh \eta \cosh \omega \{\eta^3(\omega^2 + \lambda^2) + \eta \omega^2(\eta^2 + \lambda^2)\} +
+ \sinh \eta \sinh \omega \{\eta^2 \omega(\eta^2 + \omega^2 + 2\lambda^2)\} = 0.
\]  
(30)

In the case the beam has cracks we must also take intermediate conditions into account. One can present the intermediate conditions at \( x = a_j \) \( (j = 1, ..., n) \) as
\[
v(a_j - 0) = v(a_j + 0),
\]
\[
v'(a_j - 0) = v'(a_j + 0) - k_1 v''(a_j + 0),
\]
\[
\frac{v''(a_j - 0)}{\lambda_{j-1}^2} = \frac{v''(a_j + 0)}{\lambda_j^2},
\]
\[
\frac{v''(a_j - 0)}{\lambda_{j-1}^2} = \frac{v''(a_j + 0)}{\lambda_j^2},
\]
where \(k_1 = 6\pi h_j f(s_j)\). Here \(f(s_j)\) is the stress correction function known from the linear elastic fracture mechanics (see Anderson [1]) and \(s_j = c_j/h_0\).

Now let us present the systems of boundary and intermediate conditions for the beams with one crack for the three different cases already considered above. Here \(\alpha = a/l\).

In case of \(\varphi_j > 1\):
\[
A_{11}s_{11}^2 \cos s_{11} + B_{11}s_{11}^2 \sin s_{11} + C_{11}s_{21}^2 \cos s_{21} + D_{11}s_{21}^2 \sin s_{21} = 0,
\]
\[
A_{11}s_{11}(s_{11}^2 - \lambda^2) \sin s_{11} - B_{11}s_{11}(s_{11}^2 - \lambda^2) \cos s_{11} +
+C_{11}s_{21}(s_{21}^2 - \lambda^2) \sin s_{21} - D_{11}s_{21}(s_{21}^2 - \lambda^2) \cos s_{21} = 0,
\]
\[
A_{10}(\cos s_{10}\alpha - \cos s_{20}\alpha) + B_{10} \left( \sin s_{10}\alpha - \frac{s_{10}}{s_{20}} \sin s_{20}\alpha \right) -
-A_{11} \cos s_{11}\alpha - B_{11} \sin s_{11}\alpha - C_{11} \cos s_{21}\alpha - D_{11} \sin s_{21}\alpha = 0,
\]
\[
A_{10}(s_{10}\sin s_{10}\alpha - s_{20}\sin s_{20}\alpha) - B_{10}(s_{10}\cos s_{10}\alpha - s_{20}\cos s_{20}\alpha) +
A_{11}(k_1s_{11}^2 \cos s_{11}\alpha - s_{11}\sin s_{11}\alpha) + B_{11}(k_1s_{11}^2 \sin s_{11}\alpha + s_{11}\cos s_{11}\alpha) +
C_{11}(k_1s_{21}^2 \cos s_{21}\alpha - s_{21}\sin s_{21}\alpha) + D_{11}(k_1s_{21}^2 \sin s_{21}\alpha + s_{21}\cos s_{21}\alpha) = 0,
\]
\[
A_{10}(s_{10}\cos s_{10}\alpha - s_{20}\cos s_{20}\alpha) + B_{10}(s_{10}\sin s_{10}\alpha - s_{20}\sin s_{20}\alpha) -
-A_{11} s_{11}\sin s_{11}\alpha - B_{11} s_{11}\cos s_{11}\alpha - C_{11} s_{21}\cos s_{21}\alpha - D_{11} s_{21}\sin s_{21}\alpha = 0,
\]
\[
A_{10}(s_{10}\sin s_{10}\alpha - s_{20}\sin s_{20}\alpha) - B_{10}(s_{10}\cos s_{10}\alpha - s_{20}\cos s_{20}\alpha) -
-A_{11} s_{11}\sin s_{11}\alpha + B_{11} s_{11}\cos s_{11}\alpha - C_{11} s_{21}\sin s_{21}\alpha + D_{11} s_{21}\cos s_{21}\alpha = 0,
\]
if \(\varphi_j = 1\):
\[
A_{21}s_{31}^2 \cos s_{31} + B_{21}s_{31}^2 \sin s_{31} + C_{21}s_{31}(s_{31} \cos s_{31} + 2 \sin s_{31}) +
+D_{21}s_{31}(s_{31} \sin s_{31} - 2 \cos s_{31}) = 0,
\]
\[
A_{21}s_{31}(s_{31}^2 - \lambda^2) \sin s_{31} - B_{21}s_{31}(s_{31}^2 - \lambda^2) \cos s_{31} +
+C_{21}[s_{31}(s_{31}^2 - \lambda^2) \sin s_{31} - (3s_{31}^2 - \lambda^2) \cos s_{31}] -
-D_{21}[s_{31}(s_{31}^2 - \lambda^2) \cos s_{31} + (3s_{31}^2 - \lambda^2) \sin s_{31}] = 0,
\]
\[
B_{20}(\sin s_{30}\alpha - \alpha s_{30}\cos s_{30}\alpha) + D_{20}(\sin s_{30}\alpha - A_{21}\alpha \cos s_{31}\alpha -
-B_{21} \sin s_{31}\alpha - C_{21} \alpha \cos s_{31}\alpha - D_{21} \alpha \sin s_{31}\alpha = 0,
\]
\[
B_{20} \alpha s_{30}^2 \sin s_{30}\alpha + D_{20}(\sin s_{30}\alpha + \alpha s_{30}\cos s_{30}\alpha) -
-A_{21}(k_1s_{31}^2 \cos s_{31}\alpha - s_{31}\sin s_{31}\alpha) - B_{21}(k_1s_{31}^2 \sin s_{31}\alpha + s_{31}\cos s_{31}\alpha) -
-C_{21}[\alpha(k_1s_{31}^2 \cos s_{31}\alpha - s_{31}\sin s_{31}\alpha) + (2k_1s_{31} \sin s_{31}\alpha + \cos s_{31}\alpha)] -
-D_{21}[\alpha(k_1s_{31}^2 \sin s_{31}\alpha + s_{31}\cos s_{31}\alpha) - (2k_1s_{31} \cos s_{31}\alpha - \sin s_{31}\alpha)] = 0,
\]
\[
B_{20}(s_{30}^2 \sin s_{30}\alpha + \alpha s_{30}^2 \cos s_{30}\alpha) + D_{20}(2s_{30} \cos s_{30}\alpha - \alpha s_{30}^2 \sin s_{30}\alpha) +
\]
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+\(A_1 s_{31}^2 \cos s_{31} \alpha + B_1 s_{31}^2 \sin s_{31} \alpha + C_1 (2 s_{31} \sin s_{31} \alpha + a s_{31}^2 \cos s_{31} \alpha) - D_1 (2 s_{31} \cos s_{31} \alpha - a s_{31}^2 \sin s_{31} \alpha) = 0,
\)
\(B_2 (2 s_{30}^3 \cos s_{30} \alpha - a s_{30}^4 \sin s_{30} \alpha) - D_2 (3 s_{30}^2 \sin s_{30} \alpha + a s_{30}^3 \cos s_{30} \alpha) - A_2 s_{31}^3 \sin s_{31} \alpha + B_2 s_{31}^3 \cos s_{31} \alpha + C_2 (3 s_{31}^2 \cos s_{31} \alpha - a s_{31}^3 \sin s_{31} \alpha) + D_2 (3 s_{31}^2 \sin s_{31} \alpha + a s_{31}^3 \cos s_{31} \alpha) = 0
\)

and if \(\varphi_j < 1:\)
\(A_3 \eta_j^2 \cosh \eta_j + B_3 \omega_j^2 \cosh \omega_j + C_3 \eta_j^2 \sinh \eta_j + D_3 \omega_j^2 \sinh \omega_j = 0,
\)
\(A_3 \eta_j^2 (\eta_j^2 + \lambda^2) \sinh \eta_j + B_3 \omega_j (\omega_j^2 + \lambda^2) \sinh \omega_j + C_3 \eta_j (\eta_j^2 + \lambda^2) \cosh \eta_j + D_3 \omega_j (\omega_j^2 + \lambda^2) \cosh \omega_j = 0,
\)
\(A_3 (\cosh \eta_0 \alpha - \cosh \omega_0 \alpha) + C_3 \left( \sinh \eta_0 \alpha - \frac{\eta_0}{\omega_0} \sinh \omega_0 \alpha \right) - A_3 \cosh \eta_1 \alpha - B_3 \cosh \omega_1 \alpha - C_3 \sinh \eta_1 \alpha - D_3 \sinh \omega_1 \alpha = 0,
\)
\(A_3 (\eta_0 \sinh \eta_0 \alpha - \omega_0 \sinh \omega_0 \alpha) + C_3 (\eta_0 \cosh \eta_0 \alpha - \eta_0 \cosh \omega_0 \alpha) + A_3 (k_1 \eta_1^2 \cosh \eta_1 \alpha - \eta_1 \sinh \eta_1 \alpha) + B_3 (k_1 \omega_1^2 \cosh \omega_1 \alpha - \omega_1 \sinh \omega_1 \alpha) + C_3 (k_1 \eta_1^2 \sinh \eta_1 \alpha - \eta_1 \cosh \eta_1 \alpha) + D_3 (k_1 \omega_1^2 \sinh \omega_1 \alpha - \omega_1 \cosh \omega_1 \alpha) = 0,
\)
\(A_3 (\eta_0 \cosh \eta_0 \alpha - \omega_0 \cosh \omega_0 \alpha) + C_3 (\eta_0 \sinh \eta_0 \alpha - \eta_0 \omega_0 \sinh \omega_0 \alpha) - A_3 (k_1 \eta_1^2 \cosh \eta_1 \alpha - B_3 \omega_1^2 \cosh \omega_1 \alpha - C_3 \eta_1^2 \sinh \eta_1 \alpha - D_3 \omega_1^2 \sinh \omega_1 \alpha = 0,
\)
\(A_3 (\eta_0 \sinh \eta_0 \alpha - \omega_0^3 \sinh \omega_0 \alpha) + C_3 (\eta_0^3 \cosh \eta_0 \alpha - \eta_0 \omega_0^2 \cosh \omega_0 \alpha) - A_3 (k_1 \eta_1^3 \sinh \eta_1 \alpha - B_3 \omega_1^3 \sinh \omega_1 \alpha - C_3 \eta_1^3 \cosh \eta_1 \alpha - D_3 \omega_1^3 \cosh \omega_1 \alpha = 0.
\)

5. Numerical results and conclusions

The calculations are implemented in the case of a beam with no cracks. The results of calculations are presented in the Fig. 2. Here the critical buckling load versus \(\beta\) is portrayed. Calculations carried out showed that the stability of the beam essentially depends on the parameters of the foundation.

![Fig. 2. Critical buckling load versus the quantity \(\beta\).](image-url)
References

Vibrations of a cracked anisotropic plate

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Abstract. In this study, free vibration analysis of cracked rectangular plates is performed. However, dynamic response of the forced vibration describes changes in crack depth and location similarly to the free vibration in which the difference between natural frequencies corresponding to a change in crack depth and location only has a minor effect.

Keywords: anisotropic material, crack, free vibrations, stepped plate.

1. Introduction

Eigenvalue problems of cracked rectangular plates as dual series equations and reduced to homogeneous Fredholm integral equations of the second kind are formulated by Stahl and Keer [1]. They applied the method to cracked plates of uniform thickness and simply supported boundary conditions. An experimental study of natural frequencies of clamped rectangular plates with straight narrow slits has been described by Maruyama and Ichinomiya [2]. They investigated experimentally the effect of length, position and inclination angle of a slit on the natural frequencies. However, before-mentioned reports clarified the vibration of a plate with penetrating crack. We investigated plates with part-through cracks at the corners of the re-entrant parts of the steps. Free vibration of orthotropic rectangular plates are investigated by Jafari and Eftekhari [3] with mixed Ritz-differential quadrature method and Paiva et al. [4] using the boundary element method. Recently, a vibration analysis of thin rectangular plate has been made by Li and Yuan [5]. They used a Green quasifunction method to obtain the free vibration problem of clamped thin plates. Natural frequencies for Lévy plate is studied by Park et al. [6] by using a harmonic response estimation method. The objective of this paper is to determine the eigenfrequencies of the plate and the sensitivity of free vibrations on the crack location and its depth, also on the geometrical parameters of the plate.

2. Problem formulation and basic equations

The plates are of piece wise constant thickness and are clamped along the four outer edges, is considered. Consider the free vibrations of thin anisotropic rectangular plates with length \( l \) and width \( b \). Assume that the thickness \( h_j = \gamma_j h_0 \), where \( 0 < \gamma_j \leq 1 \) for \( x \in (a_j, a_{j+1}) \), where \( j = 0, ..., n \). The parameters \( h_j, a_j, \gamma_j \) will be treated as given constants.

It is assumed, that the plate has part-through cracks [7] at the corners of the re-entrant parts of the steps. Hence according parameters for the cracks are the crack position \( a_j = \alpha_j l \), where \( 0 \leq \alpha_j \leq 1 \) and the crack length \( c_j = s_j h_j \), where \( 0 \leq s_j < 1 \).
The partial differential equation which governs the free transverse motion $w(x,y,t)$ of a plate is given by Reddy [8]

$$D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = q - I_0 \frac{\partial^2 w}{\partial t^2}. \quad (1)$$

In (1) $w(x,y,t)$ is the deflection of the plate, $D_{ij}$ stand for flexural stiffness coefficients, $I_0 = \rho bh^3$ is the moment of inertia, $\rho$ is the density of the material and $q$ is the load intensity. Since we study free vibrations of the plate one has to take $q = 0$.

The deflection of the plate $w(x,y,t)$ is presented in the form

$$w(x,y,t) = X(x) \sin \left(\frac{k\pi y}{b}\right) \cos(\omega t). \quad (2)$$

Substituting (2) into (1) we obtain

$$D_{11}X'''' - 2(D_{12} + 2D_{66}) \left(\frac{k\pi}{b}\right)^2 X'' + D_{22} \left(\frac{k\pi}{b}\right)^4 X - I_0 \omega^2 = 0. \quad (3)$$

If the material is a unidirectional fiber reinforced composite then the constants $D_{ij}$ take the form like in [3].

Solution of (3) can be presented as

$$w(x) = A_1 \sin \lambda_0 x + A_2 \cos \lambda_0 x + A_3 \sinh \mu_0 x + A_4 \cosh \mu_0 x, \quad x \in [0,a], \quad (4)$$

$$w(x) = B_1 \sin \lambda_1 x + B_2 \cos \lambda_1 x + B_3 \sinh \mu_1 x + B_4 \cosh \mu_1 x, \quad x \in [a,l], \quad (5)$$

where $\lambda_j$ and $\mu_j$ are roots of characteristic equations of (3). Arbitrary constants $A_i, B_i$ ($i = 1, \ldots, 4$) are integration constants, which are to be determined using the boundary requirements and intermediate conditions at $x = a$ as presented in [9], where $p$ is the anisotropic material stiffness parameter defined as follows.

However, it appears that the quantity $w'(x)$ can not be continuous at $x = a$ according to the model of distributed line springs developed by Dimarogonas [10] and for anisotropic bodies by Su and Sun [11].

Moreover, let us confine to the case of cracks of the first mode. Let the stress intensity coefficient and the bending moment applied to the crack be $K_j$ and $M_j$, respectively.

Following the paper [12] we can state that in the case of orthotropic materials

$$K_j = \frac{6M_j}{bh} \sqrt{\pi c_j} F(s_j) Y(\xi), \quad (6)$$

provided $h_j < h_{j-1}$ and $Y(\xi) = 1 + 0.1(\xi - 1) - 0.016(\xi - 1)^2 + 0.002(\xi - 1)^3$.

Here

$$\xi = \sqrt{\frac{E_{11}E_{22}}{2G_{12}}} - \nu_{12} \sqrt{\frac{E_{22}}{E_{11}}} \quad (7)$$

and $F$ is a shape function which must be determined on the basis of experimental data and it is shown in the previous studies by Lellep and Sakkov [13]. It is known in the fracture mechanics that the energy release rate; the generalized force $P$ and the compliance $C$ are coupled as (see Anderson [14])

$$G = \frac{P^2}{2b} \frac{dc}{dc}, \quad (8)$$

where
and $A$ is a parameter of anisotropic material. It was shown in the previous studies by Lellep and Sakkov [13] that the slope discontinuity can be calculated as

$$\Theta_j = C_j M_j,$$  

(10)

where $C_j$ is given by (9). And we get anisotropic material stiffness parameter

$$p = 6h J_f f(s_j) Y(\xi), \quad h_j < h_{j-1}.$$  

(11)

The obtained equation system is a linear homogeneous system of algebraic equations. It has a non-trivial solution only in the case, if its determinant $\Delta$ equals to zero. The equation $\Delta = 0$ is resolved up to the end numerically.

3. Numerical results

By solving the characteristic equations for clamped plates, frequency parameters of stepped plates with a part-through crack are obtained as a function of crack location $\alpha$, relative step height $\gamma$ and relative crack depth $s$.

Stepped cracked plates made of E-glass/epoxy are considered. The material properties are $E_1 = 60.7$ GPa, $E_2 = 24.8$ GPa, $G_{12} = 12.0$ GPa and $\nu = 0.23$ for E-glass/epoxy.

![Fig. 1. Natural frequencies vs crack location $\alpha$ if $\gamma = 0.5$.](image)

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Fig. 1 and Fig. 2 show the variation of the natural frequencies as a function of the crack length of clamped rectangular plates with a crack in two cases, $\gamma = 0.5$ and $\gamma = 0.7$. In figures various lines correspond to different crack lengths. Natural frequencies decrease with an increase in the crack length.

![Natural frequencies vs crack position $\alpha$ if $\gamma = 0.7$.](image)

4. Conclusions

In the present paper free vibration analysis of anisotropic stepped plates with cracks has been described. This developed method can be used to solve eigenfrequency problems of stepped anisotropic plates with arbitrary boundary conditions. The remarkable convergence and accuracy of the current solution have been demonstrated through the numerical examples.

Acknowledgment

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Natural vibrations of elastic arches with cracks

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Abstract. Vibrations of elastic arches are treated. It is assumed that the arches are weakened with crack-like defects which are treated as stable part-through surface cracks. An approximate method of determination of natural frequencies of elastic arches is developed.

Keywords: crack, vibration, arch.

1. Introduction

The presence of cracks and other defects in structural elements is a source of additional compliance. Dimarogonas [4], Chondros et al [3], Rizos et al [10], Kukla [5] explored the idea of an elastic spring modelling the additional flexibility due to a crack in the cases of vibrating beams and bars weakened with crack. This approach was extended to the case of elastic plate strips by Lellep and Kägo [7]. Axisymmetric vibrations of elastic circular cylindrical shells with piece wise constant thickness were treated by Lellep and Roots [8,9].

In the present paper natural vibrations of simply supported elastic arches are studied in the case of presence of crack-like defects.

2. Formulation of the problem

Let us consider natural vibrations of an elastic arch of radius \( R = \text{const} \). It is assumed that the arch is simply supported at both ends. The position of the current cross section is defined by the angle \( \varphi \) whereas the edges of the arch correspond to \( \varphi = 0 \) and \( \varphi = \beta \), respectively.

It is assumed that at \( \varphi = \alpha \) a crack-like defect is located. The defect is treated as a stable crack; no attention will be paid to its extension. Let the depth (length) of the crack be \( c \) whereas \( c < h \). It is assumed herein that the arch has rectangular cross section with dimensions \( h \) (thickness) and \( b \) (width of the arch).

The aim of the paper is to determine the frequencies of natural vibrations and to reveal the sensitivity of eigenfrequencies to crack parameters.

Note that the arch will be treated as a curved beam with the neutral curved axis lying wholly in one plane. It is assumed that the motion of every point of the neutral curve takes place in this plane only.

3. Solution of the equation of motion

In the case of vibrations of the arch in the plane of the coordinate axis the equilibrium conditions of an element result in the equations (see Soedel [11], Lellep [6]).
\[
\frac{\partial M}{\partial s} - Q = 0,
\]
\[
\frac{\partial N}{\partial s} + \frac{Q}{R} + p_s = \rho h \ddot{U},
\]
\[
\frac{\partial Q}{\partial s} - \frac{N}{R} + p_n = \rho h \ddot{W}
\]

(1)

Here dots denote the differentiation with respect to time \( t \). In (1) \( U \) and \( W \) denote the displacements in the circumferential and transverse direction, respectively, whereas \( p_s \) and \( p_n \) stand for the external loads in these directions. Here \( N \) and \( M \) are the membrane force and moment, \( Q \) denotes the shear force; \( \rho \) is the density of the material and \( h \) – the thickness of the arch.

According to the classical approach \( p_s = u = 0 \). Assuming that

\[ M(0; t) = 0, \quad M(\beta, t) = 0 \quad (2) \]

and the membrane force \( N \) vanishes at the edges of the arch it follows from (1) that \( M = -NR \) and

\[ M'' + M + R^2(p_n - \rho h \ddot{W}) = 0, \quad (3) \]

where prims denote the differentiation with respect to the angle \( \varphi \). Evidently, \( s = \varphi R \).

The Hooke’s law can be written in generalized stresses as

\[ N = \frac{Eh}{R}(U' + W), \]
\[ M = \frac{Eh^3}{12R^2}(U' + W''), \quad (4) \]

where \( E \) stands for the Young modulus. Since \( N = 0 \) the first equation in (4) yields \( U' = -W \) and thus

\[ M = -\frac{Eh^3}{12R^2}(W + W''). \quad (5) \]

In what follows we shall concentrate on free vibrations of the arch. Taking \( p = 0 \) and substituting (5) in (3) one obtains the equation

\[ \frac{Eh^3}{12R^2}(W'' + 2W''' + W) + \rho h R^2 \ddot{W} = 0. \quad (6) \]

In order to solve the linear equation (6) with partial derivatives the method of Fourier’ will be employed. According to this concept let us assume that at each time instant and each \( \varphi \in [0, \beta] \)

\[ W(\varphi, t) = w(\varphi) \sin \omega t \quad (7) \]

where \( \omega \) stands for the frequency of natural vibrations and \( w(\varphi) \) is an unknown function of the variable \( \varphi \). The substitution (7) in (6) leads to the equation

\[ w'' + 2w''' + w(1 - k^2) = 0 \quad (8) \]

where

\[ k = \sqrt{\frac{12\rho R^4 \omega^2}{Eh^2}}. \quad (9) \]
Evidently, the roots of the characteristic equation corresponding to the linear differential equation (8) are
\[ \lambda = \pm \sqrt{-1 \pm k}. \] (10)

It is reasonable to assume that \(-1 + k > 0; -1 - k < 0\). Thus the general solution of equation (8) takes the form
\[ w = C_1 \cosh \mu \varphi + C_2 \sinh \mu \varphi + C_3 \cos \nu \varphi + C_4 \sin \nu \varphi \] (11)
for \( \varphi \in [0, \alpha] \) and
\[ w = B_1 \cosh \mu \varphi + B_2 \sinh \mu \varphi + B_3 \cos \nu \varphi + B_4 \sin \nu \varphi \] (12)
for \( \varphi \in [\alpha, \beta] \). Here the notation \( \mu^2 = 1 - k, \nu^2 = 1 + k \) is used. In (11), (12) \( C_1 - C_4 \) and \( B_1 - B_4 \) denote arbitrary constants of integration which are to be determined from boundary and intermediate conditions.

Boundary conditions for an arch simply supported at both edges have the form
\[ w(0) = w''(0) = 0, \]
\[ w(\beta) = w''(\beta) = 0. \] (13)

At the internal points of the interval \((0, \beta)\) the displacement \( W \) and its derivative, also bending moment \( M \) and shear force \( Q \) must be continuous (except at \( \varphi = \alpha \)). At the cross section \( \varphi = \alpha \) one has following continuity conditions:
\[ w(\alpha-) = w(\alpha+), \]
\[ w''(\alpha-) = w''(\alpha+), \] (14)
\[ w'''(\alpha-) = w'''(\alpha+). \]

Note that the quantity \( w' \) is discontinuous at \( \varphi = \alpha \); the jump condition for it will be presented later.

4. Additional flexibility caused by the crack

It is assumed that at the cross section \( \varphi = \alpha \) a stable crack-like defect with depth \( c \) is located. Following the papers by Dimarogonas [4], Chondros, Dimarogonas, Yao [3], Rizos et al [10], Kukla [5] it is recognized that there exists a relationship between the local compliance of the arch \( C \) and the stress intensity factor \( K \) known in the linear elastic fracture mechanics. Let us denote the discontinuity of the slope of the transverse displacement at \( \varphi = \alpha \) by \( \theta \). The quantity \( \theta \) can be treated as a generalized displacement corresponding to the generalized force (bending moment) \( M = M(\alpha, t) \) one can write
\[ \theta = C \cdot M \] (15)
where \( C \) is the additional compliance caused by the crack.

On the other hand, it is known in the fracture mechanics that the energy release rate can be calculated as (here \( A = bc \))
\[ G = \frac{M^2}{2} \frac{dC}{dA} \] (16)
or as (see Anderson [1], Brock [2])
\[ G = \frac{K^2}{E'}. \] (17)

In (17) \( E' = E \) for plane stress state and \( E' = E/(1 - v^2) \) for the plane deformation state. The stress intensity factor \( K \) itself is defined as
If the cracked element is subjected to pure bending then
\[ \sigma = \frac{6M}{bh^2}. \]  
(19)

The shape function \( F \) in (18) depends on the type of a cracked element. Following Dimarogonas [5], Rizos et al [10] we take
\[ F(s) = 1.93 - 3.07s + 14.53s^2 - 25.11s^3 + 25.80s^4 \]  
(20)

where \( s = c/h \).

Combining the relations (15)–(20) after algebraic manipulations one obtains
\[ C = \frac{72\pi}{E'h^2b} f(s). \]  
(21)

The integration of (21) leads to the result
\[ C = \frac{72\pi}{E'h^2b} f(s) \]  
(22)

where
\[ f(s) = \int_0^s \xi^2 F^2(\xi) d\xi. \]  
(23)

Thus the jump of the slope can be evaluated as
\[ w'(\alpha +) - w'(\alpha -) = p(w''(\alpha) + w(\alpha)) \]  
(24)

where
\[ p = \frac{6\pi h}{R^2(1 - \nu^2)} f(s). \]  
(25)

5. Natural frequencies of a simply supported arch

The deflected shape of the arch is presented by (11), (12). For determination of arbitrary constants one can use the boundary and intermediate conditions (13), (14) and (24) with (25). The conditions (13) furnish at \( \varphi = 0 \) the equations
\[ C_1 + C_3 = 0, \]
\[ \mu^2 C_1 - \nu^2 C_3 = 0. \]  
(26)

It immediately follows from (26) that
\[ C_1 = C_3 = 0. \]  
(27)

The rest of requirements in (13), (14), (24) can be presented as the system, provided (27) is taken into account
\[ B_1 \cosh \mu \alpha + B_2 \sinh \mu \alpha + B_3 \cos \nu \alpha + B_4 \sin \nu \alpha = C_2 \sinh \mu \alpha + C_4 \sin \nu \alpha, \]
\[ B_1 \mu \sinh \mu \alpha + B_2 \mu \cosh \mu \alpha - B_3 \nu \sin \nu \alpha + B_4 \nu \cos \nu \alpha = \]
\[ = \mu C_2 \cosh \mu \alpha + \nu C_4 \cos \nu \alpha - p[C_2(1 + \mu^2) \sinh \mu \alpha - C_4(v^2 - 1) \sin \nu \alpha], \]
\[ B_1 \mu^2 \cosh \mu \alpha + B_2 \mu^2 \sinh \mu \alpha - B_3 \nu^2 \cos \nu \alpha - B_4 \nu^2 \sin \nu \alpha = \]
\[ = C_2 \mu^2 \sinh \mu \alpha - C_4 \nu^2 \sin \nu \alpha, \]
\[ B_1 \mu^3 \sinh \mu \alpha + B_2 \mu^3 \cosh \mu \alpha + B_3 v^3 \sin v \alpha - B_4 v^3 \cos v \alpha = \]
\[ = C_2 \mu^3 \cosh \mu \alpha - C_4 v^3 \cos v \alpha, \]
\[ B_1 \cosh \mu \beta + B_2 \sinh \mu \beta + B_3 \cos v \beta + B_4 \sin v \beta = 0, \]
\[ \mu^2 (B_1 \cosh \mu \beta + B_2 \sinh \mu \beta) - v^2 (B_3 \cos v \beta + B_4 \sin v \beta) = 0. \]

Evidently, the system (28) is a linear homogeneous system of algebraic equations with unknowns \( B_1 - B_4 \) and \( C_2, C_4 \). This system has a non-trivial solution if and only if its determinant \( \Delta \) equals to zero.

### 6. Numerical results and discussion

The equation \( \Delta = 0 \) is solved numerically making use of the computer code MATLAB. Calculations carried out showed that the natural frequencies depend on the geometrical dimensions of the arch, on the physical parameters of the material and on the location of the crack and on its length. Calculations have been carried out for the case when the crack has penetrated uniformly through the width of the arch.

The results of calculations are presented in Fig. 1 for the arch with dimensions \( R = 1 \text{ m}, \ b = 0.05 \text{ m}, \ h = 0.05 \text{ m} \). The arch is made of steel with \( E = 210 \text{ GPa}, \ \nu = 0.3, \ \rho = 7865 \text{ kg} \cdot \text{m}^{-3} \).

Different curves in Fig. 1 correspond to different extensions of the crack. It can be seen from Fig. 1 that the eigenfrequency has the lowest level for the arch without any defects.

![Fig. 1. Eigenfrequencies of the arch.](image-url)
7. Concluding remarks

A method for determination of eigenfrequencies of free vibrations of arches is developed. It is assumed that the arch is made of an elastic material and that it has stable crack-like defects. Making use of the basic concepts of the linear elastic fracture mechanics the additional local compliance due to the crack is evaluated through the stress intensity factor. Calculations carried out showed that the eigenfrequency of a cracked arch is always higher than that of a structure without defects.

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Asymmetric dynamic plastic behaviour of circular plates

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Abstract. Inelastic dynamic behaviour of stepped circular plates is studied. The plate is loaded by a concentrated loading applied at a point located away from the center of the plate. Theoretical predictions of residual deflections are obtained for the case of a rectangular impulse.

Keywords: circular plate, impact loading, plasticity.

1. Introduction

Since exact theoretical solutions of problems of dynamic loading of inelastic plates and shells are available in very simple cases only it is reasonable to develop approximate theoretical predictions for these problems. An approximate method of mode form motions was suggested by Symonds [8] for evaluation of residual deflections of beams made of rigid-plastic materials. The further progress is due to Jones [1], Kaliszky [2], [3] and others. In the present paper an inelastic circular plate subjected to asymmetric loading is considered.

2. Problem formulation and basic concepts

Let us consider an eccentrically loaded circular plate of piece wise constant thickness. The plate of radius $R$ is simply supported at the edge and loaded by a concentrated loading of intensity

$$\bar{P}(t) = \begin{cases} P, & 0 \leq t \leq t_1 \\ 0, & t > t_1 \end{cases},$$

where $t$ is current time and $t_1$ – a fixed moment of time. It is assumed that $t_1$ is relatively small. The concentrated loading is applied at the point $O_1$ at the distance $a$ from the center of the plate.

In the following it is reasonable to use the polar coordinates $r, \theta$ whereas the distance $r$ is measured from the point $O_1$. Let the thickness $h(r, \theta)$ of the plate be piece wise constant, e.g.

$$h = h_j$$

for $r \in (r_j, r_{j+1})$ where $j = 0, \ldots, n$ and $\theta \in (0, 2\pi)$. Here $h_j$ ($j = 0, \ldots, n$) stand for given constants and $r_j = r_j(\theta)$ are preliminarily defined smooth functions of the polar angle. The curve $r = r_{n+1}(\theta)$ is assumed to be the contour of the plate and $r_0 = 0$. Evidently, in polar coordinates the equation of the circle $r = r_\ast = r_{n+1}$ is

$$r_{n+1} = a \cos \theta + \sqrt{R^2 - a^2 \sin^2 \theta},$$

where $a$ is the distance between the origin of coordinates and the center of the plate. The aim of the paper is to define the distribution of the residual deflections. For the sake of
simplicity it is assumed that the material of the plate is an ideal plastic material; the influence of elastic deformations and strain hardening will be neglected.

It is assumed that the plate is made of a Tresca material and that the stress-strain state of the plate corresponds to the yield regime $BC$ where $M_2 = M_j$ and $0 \leq M_1 \leq M_j$ (Fig. 1).

![Fig. 1. Tresca yield condition.](image)

Here $M_1, M_2$ stand for the principal moments whereas $M_j$ is the yield moment for the region with thickness $h_j$. It is well known that $M_j = (\sigma_0 h_j^2)/4$, $\sigma_0$ being the yield moment of the material.

According to the associated flow law $\dot{\kappa}_r = 0$, $\dot{\kappa}_\theta \geq 0$ where

$$\dot{\kappa}_r = -\frac{1}{r} \frac{\partial W}{\partial r}, \dot{\kappa}_\theta = -\frac{1}{r} \frac{\partial W}{\partial \theta}.$$  

Here $W$ stands for the transverse deflection and dots denote the differentiation with respect to time $t$.

An approximate method of mode form motions will be employed in the present study (see [1]). According to this method the displacement rate can be presented in the form of a product

$$\dot{W}(r, \theta, t) = \dot{W}_0(t) f(r, \theta).$$  

The associated flow law is satisfied if one takes

$$f(r, \theta) = 1 - \frac{r}{r_*(\theta)}.$$  

### 3. Yield line fan and the external work

The yield line concept widely used in the finite element analysis was originally developed as an upper bound method of limit analysis ([4], [5], [6]). According to this concept a kinematically admissible yield line pattern may consist of a number of singular hinge lines, continuous yield line fields (fans) and rigid portions.

The plastic energy dissipated at a hinge line of length $L_j$ formed by the adjacent parts of the plate rotating around the yield line is

$$\dot{D}_i = M_j \phi_j L_j$$  

provided $\phi_j$ is the velocity of increasing the angle between the adjacent parts of the plate [7].
However, in the case when continuous fields of yield lines occur at the limit state then instead of (7) one has (see [3]–[7])

\[
\dot{D}_l = W_0 M_0 \iint_{S} \frac{1}{rr_c} \left( 1 + 2 \frac{r'^2}{r'^2} - \frac{r''}{r} \right) r \, dr \, d\theta
\]  

(8)

where primes denote the differentiation with respect to \( \theta \) and \( S \) stands for the mid-plane of the plate. It is forthwith to emphasize that (8) holds good in the case of plates of constant thickness. Here \( M_0 = (\sigma_0 h_0^2)/4 \), \( h_0 \) being the thickness of the plate. In the case of a multistepped plate the internal energy dissipation due to the continuous yield line field can be presented as

\[
\dot{D}_l = \sum_{j=0}^{n} M_j \int_{r_j}^{r_{j+1}} \int_{0}^{2\pi} \frac{1}{r_0} \left( 1 + \frac{r'^2}{r^2} \right) d\theta \, dr
\]  

(9)

because evidently

\[
\int_{0}^{2\pi} \frac{r'^2}{r^2} - \frac{r''}{r} \, d\theta = 0.
\]

The power of external forces including these caused by the inertia is

\[
\dot{D}_e = P \dot{W}_0 - \mu_0 \dot{W} \dot{W} dS,
\]  

(10)

where \( \dot{W} \) is the velocity and \( \ddot{W} \) the acceleration at each point of the plate. Here \( \mu_0 \) stands for the mass per unit area of the middle plane of the plate. Assuming that the stress profile chosen above with the velocity field (5), (6) holds good for the whole plate one can present (10) as

\[
\dot{D}_e = P \dot{W}_0 - \mu_0 \ddot{W}_0 \dot{W} \sum_{j=0}^{n} \int_{r_j}^{r_{j+1}} \int_{0}^{2\pi} h_j \left( 1 - \frac{r}{r_0} \right)^2 r \, dr \, d\theta,
\]  

(11)

where \( \mu_0 = \mu h_j \) for \( r \in (r_j, r_{j+1}); j = 0, ..., n; \) \( \mu \) being the density of the material.

4. Determination of deflections

The displacement field will be defined under the assumption that at each time instant the internal energy dissipation \( \dot{D}_l \) is equal to the power of external loads \( \dot{D}_e \). This approach was used by Jones [1], Kaliszky [2], [3] in the cases of plates with finite displacements.

4.1. The first stage of deformation \((0 \leq t \leq t_1)\)

Making use of the equality \( \dot{D}_l = \dot{D}_e \) and substituting \( \dot{D}_l, \dot{D}_e \) from (9) and (11), respectively, one obtains
\[ W_0 \sum_{j=0}^{n} M_j \int_{r_j}^{r_{j+1}} \int_{0}^{2\pi} \frac{1}{r} \left( 1 + \frac{r'^2}{r^2} \right) d\theta dr = \]

\[ = W_0 \left( P - \mu W_0 \sum_{j=0}^{n} \int_{r_j}^{r_{j+1}} h_j \left( 1 - \frac{r^2}{r_*^2} \right) r dr d\theta \right). \tag{12} \]

It easily follows from (12) that the acceleration of the apex can be defined as

\[ \mu \ddot{W}_0 = \frac{P - \sum_{j=0}^{n} \int_{0}^{2\pi} \frac{M_j}{r_*} (r_{j+1} - r_j) \left( 1 + \frac{r'^2}{r_*^2} \right) d\theta}{\sum_{j=0}^{n} \int_{0}^{2\pi} h_j A_j d\theta}, \tag{13} \]

where

\[ A_j = \frac{1}{2} \left( r_{j+1}^2 - r_j^2 \right) - \frac{2}{3} r_*^3 (r_{j+1}^2 - r_j^2) + \frac{1}{4} r_*^4 (r_{j+1}^2 - r_j^2) \tag{14} \]

for \( j = 1, \ldots, n - 1 \) whereas

\[ A_0 = 0, \]

\[ A_n = -\frac{r_n^2}{2} + \frac{2r_n^3}{3r_*} + \frac{r_n^2}{4r_*^2} - \frac{5r_*^2}{12}. \tag{15} \]

According to (13) \( \ddot{W}_0 = \text{const} \). Thus integration yields

\[ \dot{W}_0(t) = \dot{W}_0 \]

\[ W_0(t) = \frac{1}{2} \dot{W}_0 t^2 \tag{16} \]

for \( t \in [0, t_1] \).

### 4.2. The second stage of deformation \( (t_1 \leq t \leq t_2) \)

At the moment \( t = t_1 \) the loading is abruptly removed. The subsequent motion takes place due to inertia. Evidently, the acceleration \( \ddot{W}_1 \) for the second stage of deformation can be obtained from (15) taking \( P = 0 \). Taking (15) into account one can assert that \( \ddot{W}_1 = \text{const} \).

The velocities and displacements for the second stage can be found similarly to the previous case. The second stage is completed at the instant \( t = t_2 \) when the motion ceases. The maximal permanent deflection is \( W = W_1(t_2) \).

### 5. Numerical results and discussion

As an example of theoretical predictions prescribed above a circular plate with thickness \( h_0 \) and \( h_1 \) is studied in a greater detail. It is assumed that

\[ h = \begin{cases} h_0, & r \in [0, r_1] \\ h_1, & r \in [r_1, r_*] \end{cases} \]

where \( r_* \) is defined by (3) and \( r_1 = \text{const} \).

The results of calculations are presented in Fig. 2 and Fig. 3. Fig. 2 corresponds to the plate of constant thickness and Fig. 3 to the plate with two different thicknesses. In Fig. 2 \( t_1 = 0.2 \) and different curves correspond to different values of the parameter \( \alpha = a/R \).
Here $k$ stands for the ratio $k = P / P_3$ where $P_3 = 2\pi M_0$ is the statical limit load in the case when $\alpha = 0$. 

Fig. 2. Maximal permanent deflection of the plate of constant thickness.

Fig. 3. Maximal permanent deflection of the stepped plate.
Maximal residual deflections for stepped plates are depicted in Fig. 3 for $r_1 = 0.2R$. Here $a = 0.1R$ and $h_1 = 0.8h_0$. Different curves in Fig. 3 correspond to different values of the loading. It can be seen from Fig. 3 that the greater is the applied load the greater is the maximal residual deflection as might be expected. Similarly, the longer is the loading period the greater is the maximal deflection.

6. Conclusions

An approximate method of determination of maximal residual deflections of asymmetrically loaded circular plates was developed. Although the method is applied for simply supported plates loaded by concentrated forces it is applicable for clamped plates as well. Evidently, the method described above can be extended for the case of a distributed loading whereas the distribution is asymmetrical. The method can be accommodated for functionally graded materials choosing the different thicknesses and step locations commensurable to the variable thickness at mesh points.

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References

Inelastic cylindrical shells with elastic supports

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Abstract. The post-yield behaviour of circular cylindrical shells with internal supports is studied with the aid of a geometrically non-linear theory of thin shells. Optimal positions for additional supports are determined assuming that the intermediate support is a rigid support. The effectivity of the design is evaluated numerically.

Keywords: cylindrical shell, optimization, additional support, stiffener.

1. Introduction

The strength and load carrying capacity of pressure vessels can be increased with the aid of stiffeners or additional supports. It is reasonable to put these intermediate supports at positions which lead to the maximal stiffness or load carrying capacity. Optimal location of stiffeners for circular cylindrical shells was investigated by Cinquini and Kouam [2], also Lellep [4], [5]. Lellep and Paltsepp [6]–[8] developed non-linear programming methods for determination of optimal positions of rigid hoop stiffeners for inelastic circular cylindrical shells subjected to the uniformly distributed transverse pressure and to the axial dead load. In the present paper a method of optimization is developed for cylindrical shells with supports of different type.

2. Problem formulation and governing equations

Let us consider an inelastic circular cylindrical shell of radius R and length l subjected to the internal pressure loading (Fig. 1). The shell is furnished with an absolutely rigid stiffener which will be treated as an additional support. The problem consists in the determination of the best position of the additional support. The shell is made of an inelastic material which can be considered as an anisotropic composite which obeys the flow criterion presented by Lance and Robinson [3]. We assumed that the left hand edge of the shell is simply supported and the right hand end is elastically supported. We are striving to design the shell so that its flexibility is not too large. In the literature one can find various functionals which are used for evaluation of the flexibility of a structure. In the present paper it is assumed that the measure of the flexibility of the shell can be presented as the integral deflection

\[ J = \int_0^l W \, dx \quad (1) \]
where \( W \) stands for the transverse deflection of the shell. We are investigating the behaviour of the shell in the post-yield stage. Thus it is reasonable to assume that the intensity of the pressure loading satisfies the inequalities
\[
p \geq p_0, \quad p \geq p_1,
\]
where \( p_0, p_1 \) are the limit loads for the left-hand and right-hand part of the shell, respectively.

\[
\frac{d^2M}{dx^2} - N \frac{d^2W}{dx^2} + \frac{N_2}{R} - p = 0.
\]

Here \( M \) is the radial bending moment, \( N \) stands for the axial force applied at the edge of the tube whereas \( N_2 \) is the hoop force. Note that \( N \) is assumed to be given.

The strain components consistent with the equilibrium equation (3) are
\[
\varepsilon_1 = \frac{dU}{dx} + \frac{1}{2} \left( \frac{dW}{dx} \right)^2,
\]
\[
\varepsilon_2 = \frac{W}{R}, \quad \kappa_1 = \frac{d^2W}{dx^2},
\]
where \( \varepsilon_1, \varepsilon_2 \) are linear strain components and \( \kappa_1 \) – the curvature of a generator of the shell.

The material of the shell is a ductile unidirectionally reinforced material. In the present paper we shall consider the cases when fibers are directed in the axial or hoop direction, only. Let us introduce a parameter \( k \) so that \( k = 1 \) and \( k = \alpha \) correspond to the cases of circumferential and axial orientation of fibers, respectively. Here \( \alpha \) stands for the ratio of yield stresses of fibers and the matrix material.

It appears that the stress profile lies on the face
\[
N_2 = kN_0
\]

of the yield surface corresponding to the criterion of Lance and Robinson. On this plane the bending moment is constrained by the inequalities.
\[ |M| \leq \frac{\alpha}{k} M_0. \]  

(6)

In (5), (6) \( N_0 \) and \( M_0 \) denote the yield force and yield moment for the matrix material. Boundary conditions at the left hand end are

\[ W(0) = 0, \quad M(0) = 0, \]  

(7)

whereas at \( x = l \) one has (\( Q \) is the shear force)

\[ Q(l) = \lambda W(l), \quad M(l) = \mu \frac{dW}{dx}(l). \]  

(8)

The parameters \( \lambda \) and \( \mu \) denote the coefficients of elasticity of the right hand support. Together with the elastic support we study also the case of a semi-elastic support when the support impedes displacements in the radial direction but not rotations around the circle \( x = l \). In this case \( \lambda \to \infty \) and instead of (8) one has

\[ W(l) = 0, \quad M(l) = \mu \frac{dW}{dx}(l). \]  

(9)

3. Integration of governing equations

The analysis shows that the equality (5) is satisfied at each point of the shell. It is assumed that the equality in (6) takes place for \( x \in (a_0, b_0) \) and for \( x \in (a_1, b_1) \) where \( a_0 \leq b_0 < s < a_1 \leq b_1 < l \). It follows from the associated flow law that \( \varepsilon_2 \geq 0 \) for \( x \in (0, l) \) and \( \kappa_1 = 0 \) for \( x \in (0, a_0), x \in (b_0, s), x \in (s, a_1) \) and \( x \in (b_1, l) \). This means that \( \kappa_1 \neq 0 \) only for \( x \in (a_j, b_j) \) where \( j = 0 \) and \( j = 1 \). However, if \( x \in (a_j, b_j) \) then \( M = \text{const} \) and according to (3)

\[ W'' = \frac{1}{N} \left( \frac{kN_0}{R} - P \right). \]  

(10)

Thus for \( x \in [0, s] \) the transverse deflection is defined as

\[
W = \begin{cases} 
A_0 x, & x \in [0, a_0] \\
\frac{1}{2N} \left( \frac{kN_0}{R} - P \right) x^2 + E_0 x + F_0, & x \in [a_0, b_0] \\
B_0(x - s), & x \in [b_0, s]
\end{cases}
\]  

(11)

Similarly, for \( x \in [s, l] \) one has

\[
W = \begin{cases} 
A_1 x, & x \in [0, a_1] \\
\frac{1}{2N} \left( \frac{kN_0}{R} - P \right) x^2 + E_1 x + F_1, & x \in [a_1, b_1] \\
B_1(x - s), & x \in [b_1, l]
\end{cases}
\]  

(12)

In (11), (12) \( A_j, B_j, E_j, F_j \) (where \( j = 0, 1 \)) are arbitrary constants of integration.

4. Numerical results and discussion

Numerical results are presented for the case when the left-hand edge of the tube is simply supported. However, the right-hand end of the shell has a semi-elastic support and for the intermediate support is absolutely rigid. The results are exposed in Fig. 2 and Fig. 3.
In Fig. 2 distributions of transverse deflections are presented for different values of the internal pressure. Here $k = \alpha = 1,2$.

![Fig. 2. Transverse deflections.](image)

In Fig. 3 the bending moment is portrayed. Fig. 3 corresponds to the case $k = 1, \alpha = 2$.

![Fig. 3. Bending moment.](image)

5. **Concluding remarks**

The method of optimal design of circular cylindrical shells with intermediate rigid supports is applied to the shell subjected to the internal pressure and to the axial dead load.

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Calculations carried out showed that optimal position of the support depends essentially on the stiffnesses of elastic supports.

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Elastic plastic bending of stepped circular plates

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Abstract. Axisymmetric deformations of circular plates under the transverse loading are investigated. The material of plates is assumed to be an ideal elastic plastic material obeying the diamond yield condition and the associated flow law. The stress strain states of the plates are determined for initial elastic and the subsequent elastic plastic stages of deformation.

Keywords: plate, plasticity, axisymmetric bending.

1. Introduction

Although the investigation of elastic plastic plates began several decades ago the available solutions regard to the plates of constant thickness only. The early works of this kind are published by Eason [1], Hodge [2], Tekinalp [6]. Much later Upadrasta et al [7] employed the deformation-type theory of plasticity to develop an iterative procedure for approximate solution of this problem. In the present paper the elastic plastic bending problem is solved for circular plates of piecewise constant thickness assuming the material obeys the diamond yield condition. In the previous study of authors [4] an annular plate clamped at the inner edge was investigated.

2. Preliminaries and basic hypotheses

Let us consider the quasistatic behaviour of an elastic plastic circular plate of radius $R$ under the lateral pressure of intensity $P = P(r)$, $r$ being the current radius. It is assumed that the plate is of sandwich-type consisting of carrying layers of thickness $h$ and of a core material between the rims. The thickness of the layer of the core material is $H$.

In the present paper we assume that

$$h = h_j,$$

for $r \in (a_j, a_{j+1})$; $j = 0, \ldots, n$. The quantities $a_j$, $h_j$ are treated as preliminarily known parameters. For the sake of convenience we take $a_0 = 0$; $a_{n+1} = R$.

The response of the plate to the external loading will be prescribed by the classical plate theory. The stress components contributing to the strain energy are the bending moments $M_1, M_2$ in the radial and hoop direction, respectively. Corresponding strain components are the curvatures $\kappa_1, \kappa_2$ which can be determined via the transverse deflection $W = W(r)$.

The aim of the paper is to determine the stress strain state of the plate for the initial elastic and subsequent elastic plastic stages of loading.

3. Governing equations

In the theory of thin plates displacements and changes of angles are assumed to be small. The equilibrium conditions of a plate element have the form (see Reddy [5], Ventsel and
Krauthammer [8], Yu and Zhang [9])

\[
\frac{d}{dr}(rM_1) - M_2 - rQ = 0 \quad (2)
\]

where \( Q \) is the shear force generated by the tangential stresses. At the same time the principal moments \( M_1, M_2 \) summarize the effect of normal stresses \( \sigma_r \) and \( \sigma_\theta \).

The strain components corresponding to the classical bending theory are

\[
\varepsilon_1 = -\frac{d^2W}{dr^2}, \varepsilon_2 = -\frac{1}{r} \frac{dW}{dr}. \quad (3)
\]

If the external loading is relatively low, then entire plate is in the elastic state. In the elastic stage the curvatures (3) satisfy the Hooke’s law. The latter reads for \( r \in (a_j, a_{j+1}) \) as

\[
M_1 = D_j (\varepsilon_1 + \nu \varepsilon_2)
\]

\[
M_2 = D_j (\varepsilon_2 + \nu \varepsilon_3)
\]

where

\[
D_j = \frac{Eh_jH^2}{2(1-\nu^2)}. \quad (5)
\]

Here \( E \) and \( \nu \) stand for Young and Poisson modulus, respectively.

If the load intensity is high enough plastic deformations occur in certain regions. Now the plate is divided into elastic and plastic regions. Let us denote these regions \( S_e \) and \( S_p \), respectively. In a plastic region the stress state corresponds to a point lying on the yield surface (or a yield curve).

![Fig. 1. Diamond yield condition.](image)

It is assumed that the yield condition can be presented by the diamond \( ABCD \) shown in Fig. 1. Here \( M_{0j} \) denotes the yield moment. In the case of a sandwich plate with the rim thickness \( h_j \)

\[
M_{0j} = \sigma_0 h_j H, \quad (6)
\]

\( \sigma_0 \) being the yield stress of the material. Since \( M_1 \geq 0, M_2 \geq 0 \) in the most cases one can assume that in the plastic region

\[
M_1 + M_2 = M_{0j} \quad (7)
\]
for $r \in (a_j, a_{j+1})$. Note that the diamond yield condition was suggested by Jones [3] for approximate solution of dynamic problems of plastic plates. According to the associate flow law on the side $AB$ of the diamond one has $\dot{x}_1 = \dot{x}_2$ where dots denote the differentiation with time or a time-like parameter. Making use of (3) and the deformation-type theory of plasticity the gradientality law results in the equation
\[
\frac{d^2W}{dr^2} - \frac{dW}{rdr} = 0. \tag{8}
\]

4. Integration of governing equations in elastic regions

Assume that the portion of the plate for $r \in (a_j, a_{j+1})$ is in pure elastic stress-strain state. For determination of stresses, strains and displacement one has the equations (2)–(4). Substituting (4) with the help of (3) in the equilibrium equations (2) results in
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \left( \frac{1}{r} \frac{dW}{dr} \right) \right) = \frac{P(r)}{D_j} \tag{9}
\]
for $r \in (a_j, a_{j+1})$. This is true under the condition that this interval belongs to the set $S_e$. One can easily recheck that the general solution of the equation (9) is
\[
W = \frac{Pr^4}{64D_j} + A_1j r^2 \ln r + A_2j r^2 + A_3j \ln r + A_4j \tag{10}
\]
where $A_1j - A_4j$ are arbitrary constants. Bending moments have the form
\[
M_1 = -D_j \left\{ \frac{Pr^2}{16D_j} (3 + \nu) + A_1j [2 \ln r + 3 + \nu (2 \ln r + 1)] + 2A_2j (1 + \nu) + A_3j r^2 (\nu - 1) \right\},
\]
\[
M_2 = -D_j \left\{ \frac{Pr^2}{16D_j} (1 + 3\nu) + A_1j [2 \ln r + 1 + \nu (2 \ln r + 3)] + 2A_2j (1 + \nu) + A_3j r^2 (1 - \nu) \right\}. \tag{11}
\]

5. Solution of governing equations in plastic regions

Let us consider now the case when an interval $(a_j, a_{j+1})$ where the thickness of carrying layers is $h_j$ belongs to the set $S_p$. In plastic regions one has to satisfy the equations (7), (8) and the equilibrium equations (2). Integrating (8) one easily obtains
\[
W = A_j r^2 + B_j, \tag{12}
\]
where $A_j$ and $B_j$ are arbitrary constants. For determination of stress components one can use equations (2) and (7). The second equation in the system (2) gives after integration
\[ Q = -\frac{1}{r} \int p(r)rdr + C_{1j}. \]  

(13)

It is worthwhile to mention that the shear force must be continuous for \( r \in [0,R] \). However, the function \( P = P(r) \) can be discontinuous. In the particular case when \( P(r) = \text{const} \) instead of (13) one has for \( r \in [0,R] \)

\[ Q = -\frac{pr}{2} \]  

(14)

where the symmetry condition \( Q(0) = 0 \) has taken into account.

Substitution of (7) and (13) in (2) leads to the linear differential equation

\[ \frac{dM_1}{dr} + \frac{2}{r} M_1 = \frac{M_{0j}}{r} - \frac{1}{r} \int Prdr + C_{1j}. \]  

(15)

for \( r \in (a_j, a_{j+1}) \).

In order to find the general solution for (15) let us first consider the corresponding homogeneous equation.

Evidently, the general solution of it has the form

\[ M_h = \frac{C}{r^2}. \]  

(16)

The method of variation of the constant in (15), (16) yields

\[ C(r) = M_{0j} \frac{r^2}{2} - \frac{1}{r^2} \int (r \int Prdr) dr + C_{1j} \frac{r^3}{3} + C_{2j}. \]  

(17)

Thus the radial bending moment in a plastic region \( (a_j, a_{j+1}) \) is defined as

\[ M_1 = \frac{1}{2} M_{0j} - \frac{1}{r^2} \int (r \int Prdr) dr + C_{1j} \frac{r^3}{3} + C_{2j} \frac{r^2}{r^2}. \]  

(18)

Integration constants \( C_{1j}, C_{2j} \) in (18) can be determined using the continuity requirements of \( M_1 \) and the boundary conditions. If, for instance, the plastic region is located near the center of the plate for \( r \in [0,\eta R] \) where \( \eta < 1 \) then evidently \( C_{20} = 0 \). Otherwise the moment \( M_1 \) is not limited. Thus in this case

\[ M_1 = \frac{1}{2} M_{00} - \frac{pr^2}{8}. \]  

(19)

Here \( M_{00} \) denotes the limit moment for the portion of the plate with thickness \( h_0 \). The circumferential moment can be found according to (7), (19) as

\[ M_2 = \frac{1}{2} M_{00} + \frac{pr^2}{8}. \]  

(20)

6. Simply supported plate with a single step

Consider now a particular case of the problem posed above when \( n = 1 \) and the thickness distribution is

\[ h = \begin{cases} h_0, & r \in [0,a], \\ h_1, & r \in [a,R]. \end{cases} \]  

(21)

Let the applied loading be of constant intensity.
For the concreteness sake let us assume that the plate is simply supported at the edge. Thus at the boundary of the plate
\[ M_1(R) = 0, W(R) = 0. \] (22)
It is reasonable to assume that the deformation process consists of the initial elastic and subsequent elastic plastic stages.

6.1. Elastic stage of deformation

In the case of smaller values of the intensity of the transverse pressure the plate remains elastic. In the elastic stage the stress strain state of the plate is defined by (10) and (11). Here one has to take \( j = 0 \), if \( r \in [0, a] \) and \( j = 1 \), if \( r \in [a, R] \).

Calculating the shear force for the elastic plate one reaches to the relations (13) and in the case of constant loading to (14). On the other hand
\[ Q = \frac{1}{r} \left( \frac{d}{dr} (rM_1) - M_2 \right). \] (23)
Substituting \( M_1, M_2 \) from (11) in (23) and comparing the result with (14) one can see that \( A_{1j} = 0 \).

Evidently, \( M_1(0) = 0 \) must be finite. Thus it follows from (10) that \( A_{30} = 0 \). For determination of the rest unknown constants \( A_{20}, A_{40}, A_{21}, A_{31}, A_{41} \) one can use the boundary conditions (22) and the continuity requirements
\[ [W(a)] = 0, [M_1(a)] = 0, \left[ \frac{d}{dr} W(a) \right] = 0. \] (24)
Here square brackets denote finite jumps of corresponding variables at the given points.

The elastic state of the plate endures until the stress profile in the plane of the principal moments reaches the yield curve. This happens at the loading level when the radial moment at the centre of the plate attains the limit value \( M_{00}/2 \).

6.2. Elastic plastic stage of deformation

During the subsequent increasing of the load intensity the plate is divided into elastic and plastic regions. Plastic deformations occur in the central part of the plate with radius \( \eta R \). However, the outward part of the plate remains elastic.

In the central plastic region the stress profile lies on the yield curve; principal moments are defined by (19), (20), provided \( a \) is relatively big. Of course at the same time \( a < R \).

In the case of small values of \( a \) the quantity \( \eta R \) can be bigger than \( a \). In this case the radial bending moment \( M_1 \) is defined by (18) separately for both regions.

7. Numerical results and discussion

Results of calculations for plates with thicknesses \( h_0, h_1 \) are presented in Fig. 2–4. The distributions of transverse deflections of the plate of constant thickness are presented in Fig. 2. Different curves in Fig. 2 correspond to different values of the transverse pressure. Here \( R = 0.5, H = 0.025 \). It can be seen from Fig. 2 that the deflections at each point of the plate increase with the pressure loading, as might be expected. The load-deflection relations are depicted in Fig. 3 for different values of the thickness \( \gamma = h/h_*, h_* \) being a constant. Here \( w_0 \) stands for the deflection at the centre of the plate.
Distributions of bending moment $m_2$ are shown in Fig. 4. Here

$$m_2 = \frac{M_2}{M_{00}},$$

$M_{00}$ being the limit moment for the internal part of the plate. Fig. 4 reveals the obstacle that the hoop moment is discontinuous at circles where the thickness has abrupt changes. Moreover, the function $m_2(r)$ is not a smooth one. It has discontinuities of the derivative at the boundaries between elastic and plastic regions.

8. Concluding remarks

A method of analysis of elastic plastic circular plates with piecewise constant thickness was developed in the case of materials obeying the generalized diamond yield condition and the associated gradientality rule. The deformation-type theory of plasticity was used.
The exact theoretical solutions deduced in this study for the diamond yield conditions can be considered as approximate theoretical predictions for plates made of Tresca materials. Numerical analysis showed that the approximate theoretical results are quite close to the exact results.

Fig. 4. Circumferential moments.

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Finite element analysis of ring-stiffened conical shells

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Abstract. Ring-stiffened elastic conical shells subjected to the distributed transverse pressure are studied. Making use of the linear approximation for displacements in the longitudinal direction and the third order approximation in the radial direction a finite element model is developed for axisymmetric conical shells of piece wise constant thickness.

Keywords: thin walled shell, elasticity, ring-stiffened conical shell, finite elements.

1. Introduction

Thin walled conical shells are widely used in the machinery. Optimal designs of conical shells made of elastic and inelastic materials are established by Lellep and Puman [3]. In [4] minimum weight designs are established for inelastic circular conical shells. An iterative method for optimization of elastic circular plates was developed by Lamblin, Guerlement, Cinquini [2]. In this paper the plate is divided into axisymmetric elements, each element being a ring of constant thickness. Resorting to the principle of virtual work the stiffness matrix of an element is deduced. In the present paper the ideas of the paper [2] are extended to the case of conical shells of piece wise constant thickness subjected to the lateral loading.

2. Problem formulation

Let us study the behaviour of an axisymmetric conical shell (Fig. 1) subjected to an axisymmetric loading. Let the inner radius of the mid surface be \( a \) and the outer radius \( R \), respectively. The thickness of shell wall is \( h_1 \) and the thickness of ring-stiffened part is \( h_2 \).

\[
h = \begin{cases} h_1, & r \in [a, a_1] \\ h_2, & r \in [a_1, a_2] \\ h_1, & r \in [a_2, R] \end{cases}
\]  

(1)

Fig.1. Ring-stiffened conical shell.
The outer edge is clamped and the inner edge is free. The shell is loaded by a uniformly distributed transverse pressure of intensity $P$.

In the case of a conical element the strain components have the form [1], [4]

$$
\varepsilon_1 = \frac{dU}{dr} \cos \varphi ,
\varepsilon_2 = \frac{1}{r} (U \cos \varphi + W \sin \varphi ) ,
$$

$$
\kappa_1 = - \frac{d^2 W}{dr^2} \cos^2 \varphi ,
\kappa_2 = - \frac{1}{r} \frac{dW}{dr} \cos^2 \varphi ,
$$

(2)

where $\varepsilon_1, \varepsilon_2$ stand for linear extension ratios and $\kappa_1, \kappa_2$ are curvatures of the middle surface of the shell, $W$ and $U$ stand for displacements in the two orthogonal directions and $\varphi$ stands for the angle of inclination of the middle surface.

In what follows we will treat the shells made of elastic materials. It is well known that in the case of an elastic material the Hooke’s law holds good. The latter can be presented as (see Hodge [1], Ventsel and Krauthammer [7])

$$
N_1 = D_{0j}(\varepsilon_1 + \nu \varepsilon_2),
N_2 = D_{0j}(\varepsilon_2 + \nu \varepsilon_1),
M_1 = D_j(\kappa_1 + \nu \kappa_2),
M_2 = D_j(\kappa_2 + \nu \kappa_1),
$$

(3)

for $r \in [a_j, a_{j+1}], j = 0, 1, 2$, where $\nu$ stands for the Poisson’s modulus and

$$
D_j = \frac{E h_j^3}{2(1-\nu^2)}, D_{0j} = \frac{2E h_j}{1-\nu^2} .
$$

(4)

In (3) $N_1, N_2$ are the membrane forces and $M_1, M_2$ bending moments in the radial and circumferential direction, respectively. Here $E$ denotes the Young modulus.

At the outer edge of the shell following boundary conditions must be satisfied by the stresses

$$
M_1(R) = 0, N_1(R) = 0 .
$$

(5)

At the inner edge of the shell the boundary conditions are

$$
M_1(a) = 0, Q(a) = 0 ,
$$

(6)

where $Q$ stands for the shear force. The kinematical boundary conditions are

$$
W(R) = 0 , W'(R) = 0 .
$$

(7)

3. Finite element model of the shell

Let us consider an axisymmetric elastic conical shell. Assume that the shell is divided into elements so that in each element between $r = a$ and $r = b$

$$
w = C_1 + C_2 r + C_3 r^2 + C_4 r^3
$$

(8)

and

$$
u = A_1 + A_2 r .
$$

(9)
Note that the approximation (8) corresponds to beam elements [5]. Let us denote the vector of node values as
\[ \{v\} = [u(a) \ w(a) \ w'(a) \ u(b) \ w(b) \ w'(b)]^T \]  
(10)
and shape functions as
\[ \{N\} = [1 \ r \ 1 \ r^2 \ r^3]^T, \]  
(11)
where the superscript $^T$ denotes the transposition. Let
\[ \{q\} = [A_1 \ A_2 \ C_1 \ C_2 \ C_3 \ C_4]^T. \]
Evidently
\[ \{v\} = [B]\{q\}, \]  
(12)
where
\[
[B] = \begin{bmatrix}
1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & a & a^2 & a^3 \\
0 & 0 & 0 & 1 & 2a & 3a^2 \\
1 & b & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & b & b^2 & b^3 \\
0 & 0 & 0 & 1 & 2b & 3b^2
\end{bmatrix}
\]  
(13)
Let us denote
\[ \{\kappa\} = [P_q]\{q\}, \]  
(14)
where \(\{\kappa\} = [\kappa_1 \ \kappa_2 \ \kappa_3 \ \kappa_4]\) and
\[
[P_q] = \begin{bmatrix}
0 & \cos \varphi & 0 & 0 & 0 & 0 \\
\cos \varphi & \cos \varphi & \sin \varphi & r \sin \varphi & r^2 \sin \varphi & \\
0 & 0 & 0 & 0 & -2 \cos^2 \varphi & -6r \cos^2 \varphi \\
0 & 0 & 0 & -\frac{1}{r} \cos^2 \varphi & -2 \cos^2 \varphi & -3r \cos^2 \varphi
\end{bmatrix}
\]  
(15)
Making use of the matrix $[B]$, one can present the equation for determination of node values as
\[ [K]\{v\} = \{f\}. \]  
(16)
In (16) the stiffness matrix $[K]$ can be calculated as
\[ [K] = \int_a^b [P]^T [D][P] r \ dr, \]  
(17)
where $[P] = [P_q][B]^{-1}$. The load vector $\{f\}$ in (16) has the form
\[ \{f\} = \int_a^b ([B]^{-1})^T \{N\} r \ dr. \]  
(18)
In (17) the matrix $[D]$ has following elements
\[ [D] = \frac{Eh}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & \frac{h^2}{12} & \frac{h^2\nu}{12} \\ 0 & 0 & \frac{h^2\nu}{12} & \frac{h^2}{12} \end{bmatrix}. \] (19)

Evidently, for an element lying between \( r = a_{i-1} \) and \( r = a_i; \ i = 1,2,\ldots,n \), where \( n \) is the number of elements and \( s = (R - a)/n; a_{i-1} = a + (i-1)s; a_i = a + i \cdot s \) instead of matrix \([B]\) one has

\[ [B]_i = \begin{bmatrix} 1 & a_{i-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{i-1} & a_{i-1}^2 & a_{i-1}^3 \\ 0 & 0 & 0 & 2a_{i-1} & 3a_{i-1}^2 \\ 1 & a_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_i^2 & a_i^3 \\ 0 & 0 & 0 & 2a_i & 3a_i^2 \end{bmatrix}. \] (20)

Now the vector of node values is

\[ \{v_i\} = [u(a_{i-1})\ w(a_{i-1})\ w'(a_{i-1})\ u(a_i)\ w(a_i)\ w'(a_i)]^T \] (21)

The stiffness matrix of an element can be calculated by (17), provided \([B]^{-1}\) is substituted by \([B]_i^{-1}\) and the integral is calculated over the area of the element.

Note that the global stiffness matrix is compiled in the traditional manner (see Ottosen, Petersson [5]).

It is easy to show that in the case of a perfect plastic material the matrix \([D]^{-1}\) in (17) must be modified appropriately.

4. Numerical results and discussion

Numerical results are presented for conical shells of constant thickness and ring-stiffened shells with different locations of stiffeners. Here following notations are used:

\[ k = \frac{M_0}{RN_0 \sin \varphi} = 0.3; \ P = 5 \cdot 10^{10} Pa = 50 \text{ GPa}, \]

where \( N_0 = 2\sigma_0 h \) and \( M_0 = \sigma_0 h^2 \); \( \sigma_0 \) is the yield stress of the material. The shell is clamped at the outer edge and it is absolutely free at the inner edge. The dimensions of the shell are: \( a = 10 \text{ cm}, \ \varphi = 16^\circ, h_1 = 2.5 \text{ cm}, \ h_2 = 5 \text{ cm}, R = 1 \text{ m} \).

The elastic moduli of the material are \( \nu = 0.3 \) and \( E = 210 \text{ GPa} \). In Fig. 2 and Fig. 3 the distributions of transverse deflections are presented in the cases of constant thickness and piece wise continuous thickness with stiffeners. It is assumed herein that the stiffeners are located between \( r = a_1 \) and \( r = a_2 \). In Fig. 2 the black curve corresponds to the case \( a_1 = 0.475; \ a_2 = 0.625 \) and the grey one to the case \( a_1 = 0.4; \ a_2 = 0.55 \). The dashed line in Fig. 2 and in Fig. 3 is associated with the shell without any stiffener.

In Fig. 3 line 1 corresponds to the stiffener with \( a_1 = 0.1; \ a_2 = 0.2; \) line 2–4 are associated with stiffeners lying in the intervals \([0,4; 0,55] \); \([0,475; 0,625]\) and \([0,525; 0,675]\), respectively.
5. Concluding remarks

A finite element model for the calculation of conical shells of piece wise constant thickness is developed. The method is applied for conical shells with stiffeners. Calculations carried out showed that the displacements are sensitive with respect to the location and thickness of stiffeners. It is somewhat surprising that the stiffeners located near the free edge have relatively small influence on the transverse deflection of the shell.
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References

Elastic plastic elliptical plates

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Abstract. Methods of determination of the stress strain state of elliptical plates subjected to the distributed transverse pressure are developed. The material of the plate is an ideal elastic plastic material obeying a non-linear yield condition and the associated gradientality law.

Keywords: plate, piece wise constant thickness, plasticity.

1. Introduction

Thin-walled plates and shells have many applications not only in the aircraft and car industry but in various fields of technology. There exists a long list of solutions of elastic plate problems (see Reddy [6], Ventsel and Krauthammer [8]). The available elastic plastic solutions regard mainly to the case of circular and annular plates (see Hodge [3], Tekinalp [7], Eason [1]). In the previous papers by the authors [4], [5] minimum weight designs of axisymmetric plates are developed for the von Mises, Hill’s and Tsai-Wu materials. In the current paper these methods of analysis and optimization are extended to elliptical plates of piece wise constant thickness.

2. Formulation of the problem

Let us consider the elastic plastic bending of an elliptical plate subjected to the transverse loading of intensity $P$. Let the middle plane of the plate be surrounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $x, y$ are the Cartesian coordinates and $a, b$ – semi-axes of the ellipse. In the polar coordinates $0r\Theta$ the boundary of the plate can be defined as $r = r_*(\Theta)$ where

$$r_* = \frac{ab}{\sqrt{a^2 \sin^2 \Theta + b^2 \cos^2 \Theta}}$$

and

$$x = r \cos \Theta, \quad y = r \sin \Theta.$$  

It is assumed that the plate is a sandwich plate with two carrying layers of thickness $h = h(r, \Theta)$ and with a layer of core material between the rims. Let the thickness of the core material be $H = \text{const}$. The thickness of carrying layers is assumed to be piece wise constant. Thus,

$$h(r, \Theta) = h_j$$

for $(r, \Theta) \in S_j$, where $j = 0, 1, ..., n$ and
\[ \bigcup_{j=1}^{n} S_j = S. \]  

(5)

Here \( S \) denotes the set of points surrounded by the ellipse (1). Let us denote the areas of \( S_j \), \( S \) by \( \bar{S}_j \) and \( \bar{S} \).

In the following we will investigate the direct problems of determination of the stress-strain state of the plate corresponding to the given loading level as well as problems of optimization. In the latter case we will define the plates of minimum weight for constrained displacements among other plates of piece wise constant thickness.

As we are interested in the minimization of the amount of the material of carrying layers the functional subjected to the minimization is presented as

\[ V = \sum_{j=0}^{n} h_j \bar{S}_j. \]

(6)

When minimizing the cost criterion (6) one has to impose constraints on the stress strain state of the plate. In the present paper it is assumed that the inequality

\[ W(r, \Theta) \leq W_0 \]

(7)

is satisfied for each \((r, \Theta)\) from the ellipse. Here \( W \) stands for the transverse deflection and \( W_0 \) is a given constant.

Material of the plate is assumed to be an ideal elastic plastic material obeying a non-linear yield condition of von Mises, Hill [2] or Tsai-Wu [9].

3. Governing equations

Accounting for the membrane forces \( N_r, N_\Theta, N_{r\Theta} \) and bending moments \( M_r, M_\Theta, M_{r\Theta} \) one obtains the equilibrium equations in the form (see Reddy [6], Yu and Zhan [10])

\[
\begin{align*}
\frac{\partial}{\partial r} \left( r N_r \right) + \frac{\partial N_{r\Theta}}{\partial \Theta} = & -N_\Theta = 0, \\
\frac{\partial}{\partial r} \left( r N_{r\Theta} \right) + \frac{\partial N_\Theta}{\partial \Theta} = & +N_{r\Theta} = 0, \\
\frac{\partial^2}{\partial r^2} \left( r M_r \right) - \frac{\partial M_\Theta}{\partial r} + & \frac{1}{r} \frac{\partial^2 M_\Theta}{\partial \Theta^2} + 2 \frac{\partial^2 M_{r\Theta}}{\partial r \partial \Theta} + 2 \frac{\partial M_{r\Theta}}{\partial \Theta} + Pr + \\
& + \frac{\partial}{\partial r} \left( r N_r \frac{\partial W}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \Theta} \left( N_\Theta \frac{\partial W}{\partial \Theta} \right) = 0.
\end{align*}
\]

(8)

In the case of small deformations and small displacements one can neglect the membrane forces. In this case the system (8) reduces to the equation

\[
\frac{\partial^2}{\partial r^2} \left( r M_r \right) - \frac{\partial M_\Theta}{\partial r} + \frac{1}{r} \frac{\partial^2 M_\Theta}{\partial \Theta^2} + 2 \frac{\partial^2 M_{r\Theta}}{\partial r \partial \Theta} + 2 \frac{\partial M_{r\Theta}}{\partial \Theta} + Pr = 0.
\]

(9)

The curvature components corresponding to (9) are

\[ \kappa_r = -\frac{\partial^2 W}{\partial r^2}, \]

\[ \kappa_\Theta = -\frac{1}{r} \left( \frac{\partial W}{\partial r} + \frac{1}{r} \frac{\partial^2 W}{\partial \Theta^2} \right), \]

(10)
\[ \kappa_{r\theta} = -\frac{1}{r} \left( \frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial W}{\partial \theta} \right). \]

In the elastic regions the Hooke’s law holds good. The generalized Hooke’s law can be presented as
\begin{align*}
M_r &= D_j (\kappa_r + \nu \kappa_{\theta}), \\
M_{\theta} &= D_j (\kappa_{\theta} + \nu \kappa_r), \\
M_{r\theta} &= D_j \kappa_{r\theta} (1 - \nu)
\end{align*}

for \((r, \theta) \in S_j \,(j = 0, \ldots, n)\) and
\[ D_j = \frac{E h_j H^2}{2(1 - \nu^2)}. \]

Here \(E\) and \(\nu\) stand for the Young and Poisson modulus, respectively.

The shear forces can be determined as
\begin{align*}
\Theta_r &= \frac{1}{r} \left( \frac{\partial}{\partial r} (r M_r) + \frac{\partial M_{r\theta}}{\partial \theta} - M_{\theta} \right), \\
\Theta_{\theta} &= \frac{1}{r} \left( \frac{\partial}{\partial r} (r M_{r\theta}) + \frac{\partial M_{\theta}}{\partial \theta} - M_{r\theta} \right).
\end{align*}

In the elastic plastic stage of deformation the plate is subdivided into elastic and plastic regions, respectively. In the plastic regions the Hooke’s law does not hold any more. It is assumed that the material obeys the yield criterion
\[ \Phi_j (M_r, M_{\theta}, M_{r\theta}, M_{0j}) \leq 0 \]

for \((r, \theta) \in S_j, \, j = 0, \ldots, n.\)

In a plastic region instead of the inequality (14) one has equality \(\Phi_j = 0.\) In elastic regions (14) is satisfied as a strict inequality. In (14)
\[ M_{0j} = \sigma_0 H h_j \]
where \(\sigma_0\) is the yield stress of the material.

According to the associated flow law the vector of strain rates is directed towards the external normal to the yield surface. Making use of the deformation-type theory of plasticity one can assert that in plastic regions
\[ \kappa_r = \lambda_j \frac{\partial \Phi_j}{\partial M_r}, \quad \kappa_{\theta} = \lambda_j \frac{\partial \Phi_j}{\partial M_{\theta}}, \quad \kappa_{r\theta} = \lambda_j \frac{\partial \Phi_j}{\partial M_{r\theta}} \]

where \(\lambda_j\) stands for a non-negative scalar multiplier. It is presumed herein that (16) is applied at the points \((r, \theta) \in S_j\) where the thickness of the plate is \(h_j.\)

In the present paper the yield surfaces corresponding to the yield criteria of von Mises, Hill and Tsai-Wu will be employed.

It is assumed that the edge of the plate is simply supported. Thus the boundary conditions have the form
\[ W|_{\Gamma} = 0, \quad M_n|_{\Gamma} = 0, \]
where \(\Gamma\) is the curve formed by the intersection of the middle plane with the edge of the plate and \(M_n\) is the bending moment normal to the edge.
4. Numerical results and discussion

For numerical solution of the basic equations the methods resorting to wavelets and the finite element method are employed.

The results of calculations are presented in Fig. 1–6 and Table 3.

In Fig. 1 the distribution of the plastic zone in the case of the plate of constant thickness is portrayed. The plastic deformations at first occur at the origin of coordinates and with the load increasing the plastic zone moves towards the edge of the plate.

The results obtained for a one-stepped plate with the thickness

\[ h = \begin{cases} h_0, & (r, \Theta) \in S_s \\ h_1, & (r, \Theta) \notin S_s \end{cases} \tag{18} \]

are presented in Fig. 2–6. Here \( S_s \) presents a circle with the center at \( x_1 = -0,541 \) and \( y_1 = -0,125 \) and radius \( R = 0,5 \). The midplane of the ellipse is surrounded by the ellipse with semiaxes \( a = 1, b = 0,5 \) and with the center at the origin of coordinates.

![Diagram](image)

Fig. 1. Distribution of the plastic zone for the plate of constant thickness.

The distributions of transverse deflections at sections parallel to the axes of coordinates \( x \), \( y \) respectively, are shown in Fig. 2 and Fig. 3. Here \( P = 3,4 \).

![Diagram](image)

Fig. 2. Deflections at sections parallel to \( x \)-axis.  Fig. 3. Deflections at sections parallel to \( y \)-axis.

The location of corresponding sections (lines 1–9 in Fig. 2 and 1–6 in Fig. 3) are shown in Tables 1 and 2.
Table 1. Numbers of curves for Fig. 2, 4.

<table>
<thead>
<tr>
<th>#</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
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<td>-0.60</td>
<td>-0.67</td>
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<td>0.80</td>
<td>0.92</td>
<td>-0.97</td>
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Table 2. Numbers of curves for Fig. 3, 5.

<table>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0.00</td>
<td>-0.24</td>
<td>0.38</td>
<td>0.24</td>
<td>-0.42</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Fig. 4. Bending moment in sections parallel to x-axis.

Fig. 5. Bending moment in sections parallel to y-axis.

The distributions of the bending moment $m_1 = M_1/M_0$ in these sections are presented in Fig. 4, 5. The asterisks in Fig. 3, 4 indicate the boundaries of plastic zones. It can be seen from Fig. 4, 5 that the curves 6–9 (Fig. 4) and 4–6 (Fig. 5) belong wholly to the elastic region. However, curves 1–5 in Fig. 4 cross both, the elastic and plastic regions. The curves 2, 3 in Fig. 4 demonstrate that the plastic deformations have reached to the region where the plate is thicker. The variation of the bending moment $M_1$ with the transverse pressure $p$ in the cross section $x = -0.661$ is portrayed in Fig. 6. Here $p$ stands for the non-dimensional pressure intensity

$$p = P \frac{M_{00}}{ab},$$

where $M_{00}$ stands for the yield moment for a section with thickness $h_0$.

The curves 1–6 in Fig. 5 correspond to the load intensities $p_1 = 3.8$; $p_2 = 3.7$; $p_3 = 3.6$; $p_4 = 3.5$; $p_5 = 3.2$; $p_6 = 2.6$. It can be seen from Fig. 5 that for $p < 3.5$ this section remains elastic and for $p > 3.5$ plastic deformations take place.

Table 3. Optimal parameters of the plate for $w_0 = 0.50$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$x$</th>
<th>$y$</th>
<th>$R$</th>
<th>$h_1$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>-0.307</td>
<td>-0.243</td>
<td>0.207</td>
<td>0.674</td>
<td>12.6%</td>
</tr>
<tr>
<td>3.2</td>
<td>-0.341</td>
<td>-0.207</td>
<td>0.244</td>
<td>0.683</td>
<td>10.4%</td>
</tr>
<tr>
<td>3.4</td>
<td>-0.379</td>
<td>-0.182</td>
<td>0.276</td>
<td>0.742</td>
<td>11.5%</td>
</tr>
<tr>
<td>3.6</td>
<td>-0.407</td>
<td>-0.153</td>
<td>0.340</td>
<td>0.788</td>
<td>12.1%</td>
</tr>
<tr>
<td>3.8</td>
<td>-0.506</td>
<td>-0.117</td>
<td>0.447</td>
<td>0.814</td>
<td>12.4%</td>
</tr>
</tbody>
</table>
5. Concluding remarks

Methods based on Haar wavelets and on FEM are developed for determination of the stress strain state of elliptical plates of piece wise constant thickness. The plates under consideration are subjected to the distributed transverse pressure and are simply supported at the edge. Calculations carried out showed that in the case of a plate of constant thickness the plastic is located near the center of the plate and it spreads towards the edge. In the case of stepped plates the circumferential bending moment is discontinuous whereas radial moment and shear forces remain continuous.

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References


Investigation of some properties of brush-plated gold and silver galvanic coatings

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Abstract. Nickel-hardened gold and silver coatings were brush-plated from a commercial SIFCO Dalic Solution (Gold Hard Alloy), Code SPS 5370, and Silver Hard Heavy Build, Code SPS 3083, on open thin-walled copper ring substrates. The surface structure and the microstructure of the cross section of the coatings of different thicknesses and deposited at various current densities, was studied by means of scanning electron microscopy (SEM, in Zeiss EVO MA-15, Germany). The surface of the coatings obtained by brush-plating was inhomogeneously fine-grained, and the proportions of single grains were variable; the coatings had a fine crystalline nano-structure caused by the high deposition current and by the short-term growth of formed grains. The results were compared with surface structure of the coatings obtained by tank-plated deposition. The magnitudes of the modulus of elasticity and of the nanohardness of the coatings were obtained by instrumented indentation using the MTS Nano Indenter XR® and the Micromaterials Nano Test system pendulum-type nanohardness tester. The measured values of the moduli of elasticity and nanohardness of gold and silver coatings were comparable.

Keywords: brush-plating, open ring substrate, gold and silver coatings, surface structure, microstructure, modulus of elasticity, nanohardness.

1. Introduction

The properties of an engineering component are to a great extent determined by its surface [1]. Surface layers can protect the component and prevent its wear and corrosion. In order to plate only small areas, for example, the case of repairing machine (e.g. sliding contact surfaces of generators), with no limitation to the size of the workpiece, brush plating is a relatively straightforward procedure. In recent time brush plating has reached industrial maturity and has gained some attention [2], [3], [4].

Brush plating is an electrodeposition technique where only selective areas can be plated, while the coated part does not have to be immersed in a plating tank, and the stylus (anode), covered with an absorbent material soaked in the electrolyte, can be moved to the substrate (cathode) or, alternatively, the substrate can be moved in relation to the fixed anode. In this case the anode is replenished with the plating solution by drops from a separatory funnel or by means of a pump and the coating is deposited at uniform rotation
speed, which guarantees relatively homogeneous temperature of the deposition process. Consequently, the stylus (anode) is always kept in motion whenever it is in contact with the work surface. Fig. 1 shows a schematic drawing of this process [5], [6]. The solutions used in brush plating must contain metal salts at high concentrations; this permits application of higher (about 100 times) current densities and guarantees a fast deposition process in comparison with tank plating. The high deposition current causes the fine crystalline structure of the brush-plated coating.

Fig. 1. Schematic presentation of the brush plating process of a coating on the open ring substrate.

Galvanic tank- and brush-plated gold and silver coatings are used for decorative and technical purposes [7]. Hard coatings for engineering use (e.g. on electric contacts and small machine parts) are typically plated on a metallic material and alloying metals or nano-particle composites are usually added to pure gold and silver (otherwise these are too soft). Considering the cost, gold is sometimes replaced by silver in manufacturing electronic components. Thicker electrodeposited silver coatings, 10 µm – 40 µm, used widely in industrial applications (high current contacts, parts of machines or instrumentation), do not typically retain mirror finish.

The presented brush-plating setup was elaborated for determination of residual stresses in coatings by the discrete curvature method. The aim of this study was investigate the surface morphology and the morphology of the cross-section of gold and silver coatings of different thicknesses, deposited at various current densities, and to assess the magnitudes of the modulus of elasticity and of the nanohardness of the coatings by nanoindentation.

2. Experimental procedure and method

Nickel-hardened gold and silver coatings were brush-plated from a commercial SIFCO Dalic Solution (Gold Hard Alloy), Code SPS 5370 – gold potassium cyanide, ethylenediamine, nickel cyanide pH 8.4, concentration of gold 100 g/l and Silver Hard Heavy Build, Code SPS 3080 – silver cyanide, ethylenediamine pH 11.6, concentration of silver 100 g/l, on open thin-walled copper ring substrates. The nickel preplate with
thickness of 0.4 μm was deposited from the electrolyte with following composition – nickel sulphate 350 g/l, formic acid 60 g/l, magnesium sulphate 10 g/l pH 1.6 [5]. The strips with dimensions 11.0 × 96.0 mm², used as the substrate, were cut from a copper plate and were rolled to the open ring. The coated surface of the substrate was polished to the roughness \( R_a = 0.064 \text{ μm} \). The preparation of the substrate for deposition and the plating technology can be found in the literature [5], [6], [8]. The velocity of the substrate was 0.39 m/s and room temperature was 21°C. The ratio of the anode surface to the surface to be coated was 1.8.

Average coating thickness was calculated, using a weight from the difference of the specimen is weight before and after deposition. The density of the gold (Hard) coatings was 17.6 g/cm³ [9].

<table>
<thead>
<tr>
<th>Coating code</th>
<th>( \text{Au15} )</th>
<th>( \text{AuNi25} )</th>
<th>( \text{Au30} )</th>
<th>( \text{Ag20} )</th>
<th>( \text{AgNi20} )</th>
<th>( \text{Ag40} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current density [A/dm²]</td>
<td>15</td>
<td>25</td>
<td>30</td>
<td>20</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>Working voltage [V]</td>
<td>4.5–5.0</td>
<td>6.5–7.0</td>
<td>7.0–7.5</td>
<td>5.0–5.5</td>
<td>5.0–5.5</td>
<td>7.0–7.5</td>
</tr>
<tr>
<td>Deposition temperature [°C]</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>27</td>
<td>31</td>
</tr>
<tr>
<td>Deposition rate [μm/min]</td>
<td>0.91</td>
<td>1.20</td>
<td>1.25</td>
<td>1.68</td>
<td>1.68</td>
<td>2.82</td>
</tr>
<tr>
<td>Thickness of coating [μm]</td>
<td>3.4</td>
<td>3.6</td>
<td>5.3</td>
<td>15.5</td>
<td>13.9</td>
<td>43.0</td>
</tr>
</tbody>
</table>

The plated rings were cut into pieces with a length of arc of 8 mm for SEM analysis and with a length of arc of 30 mm for determination of the modulus of elasticity and nanohardness. The surface structure and the cross section of the coatings were studied by means of scanning electron microscopy (SEM) Zeiss EVO MA-15.

The Oliver-Pharr method was used to determine the reduced modulus and hardness from the load displacement curves, using a tip areas function that was carefully calibrated on fused silica.

The magnitudes of the modulus of elasticity of the coatings were obtained by instrumented indentation using the MTS Nano Indenter XR® and the Micromaterials Nano Test system pendulum-type nanohardness tester according to the equation

\[
\frac{1}{E^*} = \frac{1 - \mu_2^2}{E_2} + \frac{1 - \mu'^2}{E'},
\]

where the literature based standard values \( E' = 11.43 \times 105 \text{ N/mm}^2 \) and \( \mu' = 0.07 \) are the modulus of elasticity and the Poisson’s ratio of the CVD diamond indenter, respectively [10]; \( \mu_2 = 0.44 \) is Poisson’s ratio for the coating, which is assumed to be same as that for pure gold; \( E^* \) and \( E_2 \) are the measured and the calculated moduli of elasticity of the coating materials, respectively. The measurements were conducted in a series of 30 measurements at loads of 1.2 mN and 4 mN. The calculated average values of the elastic modulus and of nanohardness are reported.

When a gold coating was a binary alloy of \( \text{Au} \) and \( \text{Ni} \) the reduced modulus of elasticity was calculated using formula [11]
\[ E_2 = E_{Au} + y, \]

where \( y = (E_{Ni} - E_{Au})(z/100) \) and \( z \) is the percentage of the alloying element.

Using the data reported in [12] for pure materials, \( E_{Au} = 77.2 \text{ GPa}, E_{Ni} = 207 \text{ GPa} \) and 4\% of nickel, we obtained for alloy gold \( E_2 = 82.4 \text{ GPa} \).

3. Results and discussion

The morphology of the gold and silver coatings is shown in Fig. 2. For comparison, the morphology of the gold and silver coatings deposited from a plating tank is also presented in Fig. 3. The gold coating (Fig. 3b)) was deposited from a non-cyanide tank by rotating the electrode. It is evident that the surface of the coatings obtained by brush-plating is inhomogeneously fine-grained, and the proportions of single grains are variable. The surface of the brush-plated coating is much finer and more compact than that of the tank-plated gold and silver coatings. The surface morphology of the coatings deposited at the higher current density is more homogeneous. The fine crystalline nano-structure of brush plated coatings is evidently caused by the high deposition current.

![Fig. 2. The SEM morphology of gold coatings: Au15 a), AuNi25 b) and Au30 c); morphology of silver coatings: Ag20 d), AgNi20 e) and Ag40 f).](image1)

![Fig. 3. The morphology of the gold a), b) and silver c) coatings electroplated in a plating tank at 0.25 A/dm² [13], 1.50 A/dm² [14], and 0.50 A/dm² [15].](image2)
The cross-section of the coatings is presented in Fig. 4. The thickness of the coatings was sufficiently uniform but the roughness $R_a$ increased up to 0.200 μm for gold and up to 0.150 μm for silver. If the coatings were thicker (prolonged plating time) their uniformity and roughness would deteriorate. The investigated silver coatings are to some degree thicker than gold the coatings (see Fig. 4c and d)).

The coatings contain carbon particles from the graphite anode (see Fig. 4b) Au20), which can have an effect on the mechanical properties of the coating material. As we can see, there are no microcracks, when copper is isostructural with gold but there is a clear interface between the coating and the substrate (after all they, are different materials). The nickel preplate of thickness 0.4 μm (as suggested) was used to hinder the diffuse of copper into gold coating.

![Image](image_url)

Fig. 4. The SEM morphology of polished cross-sections of the gold coatings: Au15 a), AuNi25 b) and Au30 c); of polished cross-sections of the silver coatings: Ag20 d), AgNi20 e) and Ag40 f).

The average nanohardness of the gold and silver coatings was 2.46 ± 1.28 GPa and 1.71 ± 0.34 GPa, respectively. In our experiment the nanohardness of the gold coatings was higher than that of the silver coatings. According to the textbook data [16], hard alloys of gold and silver have a microhardness of 1.90 GPa and 1.2 GPa, respectively. Nanohardness is to some degree higher than microhardness. The reduced modulus of elasticity for the gold coatings was 80.9 ± 22.8 GPa, which is comparable to the calculated value 82.4 GPa and for the silver coatings was 83.7 ± 10.7 GPa which is comparable to the value 76.0 GPa reported in [12]. The data for the gold coatings are showed considerable fluctuation, which depended evidently on the place of indentation. When the indenter is applied on a grain or between grains, the surface morphology of gold coatings is more pronounced.

Since indentation was within the range 125 – 230 nm for the gold coatings and within the range 115 – 300 nm for the silver coatings, which is significantly less than the thickness of the coatings, the mechanical data obtained from nanoindentation can be taken to be representative of the properties of the coatings without acceptable influence of the substrate.
4. Conclusion

Brush-plated coatings had a fine crystalline nano-structure which was caused by the high deposition current and by the short-term growth of formed grains. The cross-section micrograph of the coatings contained graphite particles deposited from the anode. The average values of the moduli of elasticity of the gold and silver coatings, obtained by instrumented indentation, were comparable and the value of the nanohardness of the gold coatings was higher than the nanohardness of the silver coatings. The average modulus of elasticity and the nanohardness of the gold coatings obtained by the experiment fluctuated to a great extent. The calculated value of the modulus of elasticity of the gold alloy coatings was close to the corresponding value obtained by the experiment.

Acknowledgements

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References


A MCDM software tool for automating the optimal design environments, with an application in shape optimization

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Abstract. Utilizing the proper decision support tools in developing an automated design environment for optimal shape design of textile composites is the focus of this work. Yet in this realm design decision-making is largely multiobjective in nature and characterized by various forms of uncertainty. The proposed multiple criteria decision-making (MCDM) software tool works on the basis of reactive search optimization methodology where the learning techniques are integrated into search heuristics for efficiently solving complex optimization problems. Visualization and rapid trade-off decisions are effectively used in the context of rapid virtual prototyping, and multiobjective decisions are made under uncertainty. Moreover the preferences, multiple software components, and the speedy adaptation of strategies are integrated through an online intelligent learning scheme.

Keywords: optimal engineering design, interactive multicriteria decision making, reactive search optimization, multiobjective optimization, textile composite, materials selection.

1. Introduction

Dealing with optimal design of textile composites, is a highly challenging task where the number of design criteria are increased and the geometry becomes way complicated. The challenges as such would present a large-scale MCDM design problem [1], [2] to deal with. Textile composite materials consist of a polymer matrix combined with textile reinforcement. Typical applications range from high performance aerospace components to structural parts of transportation industry. In fact because of the numerous advantages of composites in comparison to traditional materials there has been an increasing trend in the usage of composite materials in different industries. Former research on optimal design of textile composites [7], [21], highlighted that the ability to test preliminary designs is not economically workable and the assessment of preliminary materials systems urges the use of simulation tools. Such a strategy would improve the process of multi-criteria materials selection [24] and also can empower designers in considering the role of materials selection in design of materials and products. Jahan and Edwards believe [8] that there appears to be a simulation-based materials design revolution underway in which materials selection could be improved in order to more rapidly qualify new material designs. This would happen by shifting from costly and time-consuming physical experimentation to less costly computational modeling and design [23].

The integrated and multi-disciplinary design process of composites has been very challenging. The design process is divided into several criteria and sub-criteria, while receiving the contributions of many different departments trying to meet conflicting requirements of the design simultaneously. Consequently, an optimal design process within such complex systems is required through advanced decision-support tools that can
account for interactions and conflicts between several criteria. This leads to the need of optimizing several conflicting objectives simultaneously via reliable multicriteria decision-making models. For the optimal design of composites, with the aid of advancement of interdisciplinary and data analysis tools, a series of criteria including mechanical, electrical, chemical, cost, life cycle assessment and environmental aspects are now able to be simultaneously considered. As one of the most efficient approach, the MCDM applications can provide the ability to formulate and systematically compare different alternatives against the large sets of design criteria. However, the mechanical behavior of woven textiles during the draping process has not been yet fully integrated to the optimal design approaches of MCDM algorithms. In this case study the criteria of mechanical behavior of the woven textile during the draping and the further involved simulations and analysis are included in the process of the optimal design and decision-making. For this reason the proposed optimal design strategy has been upgraded in terms of complex geometry modeling, and integration to materials selection. Comparing material properties and selecting the most appropriate materials, help to enhance the performance of products. Therefore it is important to consider and rank all the available materials. A key objective of mechanical modeling of textiles is to define the dimensions and characteristics of a product and the materials from which it is made so that it can perform an acceptable function [19]. The area of the design decision-making for simultaneous consideration of the structural solution and materials selection, which is generally needed at the early design stage is relatively weak. Although the importance of integrating materials selection and product design has been often emphasized [20]. The designer in engineering of the optimal textile structures assume a material before optimizing the geometry or select the best material for an existing geometry of a structure, but clearly either approach does not guarantee the optimal combination of geometry and material [17]. Alternatively here the materials properties are directly transmitted to the design software package so that the effect of changing materials properties on the geometry and dimensions of a component design can be directly evaluated and ranked. At the same time the engineering designer can evaluate the effect of changing geometry and dimensions on product performance. Worth mentioning that the process of materials selection is highly dependent on data related to material properties. In fact with a large number of materials, clearly there is a need for an information-management system [18]. Therefore in the initial proposed optimal design strategy for interactive optimization and MCDM the existing drawbacks to utilizing MCDM are improved by connecting the data mining, visualization and optimization through the user interaction and decision-making. Besides the materials databases are used as materials selection systems, which are essentially developed for data storage searching. Moreover the electronic materials databases and data search software packages would help designer in this regard [24].

2. Draping

The manufacturing of woven reinforced composites requires a forming stage so called draping in which the preforms take the required shapes. The main deformation mechanisms during forming of woven reinforced composites are compression, bend, stretch, and shear which cause changes in orientation of the fibers. Since fiber reorientation influences the overall performance it would be an important factor that in the process of material selection to consider the draping along with the other criteria.
In an optimal engineering design process for the textile composites, the materials selection integrated with draping can well determine the durability, cost, and manufacturability of final products [21], [22]. The process would naturally involve the identification of multiple criteria properties of mechanical, electrical, chemical, thermal, environmental and life cycle costs of candidate materials [10]. In fact multiple criteria from different disciplines which are to be satisfied in a materials selection problem, often because of the criteria conflicts the complexities are even increased. Moreover the mechanical behavior of woven textiles during the draping process has not been yet fully integrated to the MCDM algorithms. Although many applications and algorithms of MCDM [10] have been previously presented to deal with decision conflicts often seen among design criteria in materials selection. However many drawbacks and challenges are identified associated with their applicability [23].

Fig. 1. Simulation of draping process including a combined mechanical modeling of compression, bend, stretch, and shear shown from two different draping angles.

3. Geometrical-mechanical modeling and simulation of draping

The mechanical models of draping with a much higher computation cost, comparing to the kinematic models, offer the benefit of representing the non-linear materials behavior. Moreover the mechanical simulation, as the most promising technique, gives a real-life prediction of the fiber reorientation. Beside of all presented approaches to the geometrical modeling of woven textiles so far [14], the Spline-based methods have been the most effective technique. In fact, the Spline-based geometrical representation of a real-life model of any type of the flat-shaped woven textile, are done with implementing the related computer aided geometrical design (CAGD) code. However the mathematical representation of a multiple-dome shaped woven, which is essential for draping simulation, in the practical scale, could not be computationally efficient. Therefore in order to handle the computational complexity of geometrical modeling the multiple-dome woven shapes, utilizing the NURBS-based CAGD packages are proposed. Khabazi [16] introduced generative algorithms for creating these complex geometries. His improved algorithm is capable of producing the whole mechanism of deformation with combining all details of compressed, bended stretched and sheared properties.
Fig. 2. A combination of four different simulation criteria including the compression, bend, stretch, and shear form the draping a) Geometrical modeling and simulation of the woven textiles b) Mechanical modeling of the bending; the behavior of textile under its weight is simulated by manipulating the related geometrical model within the CAGD package.

It is assumed that if the mechanical behavior of a particular woven fabric of a particular type and material is identified then the final geometrical model of the draping could be very accurately approximated. In this technique the defined mechanical mechanisms of a particular material, in this case glass fiber [14], are translated into a geometrical logic form integrated with the NURBS-based CAGD package through the process of scripting [16].

Worth mentioning that traditionally in order to include the materials property into the mechanical models of textile the outputs from finite element analysis (FEA) are utilized as inputs to MCDM in material selection. FEA allows materials property data to be transmitted directly to a design software package so that the effect of changing materials properties on the geometry and dimensions of a component design can be directly evaluated. At the same time the DM can evaluate the effect of changing geometry and dimensions on product performance [21].

Fig. 3. Geometrical modeling of double dome utilizing the Khabazi’s algorithm [16].

4. Integration the MCDM-assisted material selection with draping simulation

Recently a combined FEA-MCDM approach as a framework that links the capabilities of FEA tools to the MCDM approaches for composite structural materials selection problem [21] proposed. However due to the geometrically challenging modeling of the composite
product the draping simulation has not been considered in their work. In order to select the best material of a woven textile as well as the right angle of draping, the draping simulation needs to be carried out for a number of draping degrees for a particular material. The results of all the draping simulations of different drape angles are gathered as a data-set for consideration, in addition to already existed data-sets from the earlier case study [10], including the other criteria i.e. mechanical, electrical, chemical, cost, life cycle assessment and environmental.

5. Visualization; an effective approach to MCDM and materials selection

Visualization is an effective approach in the operations research and mathematical programming applications to explore optimal solutions, and to summarize the results into an insight, instead of numbers [11], [12]. Fortunately during past few years, it has been a huge development in combinatorial optimization, machine learning, intelligent optimization, and RSO, which have moved the research in advanced visualization methods forward [13]. The previous work in the area of visualization for MCDM [13], allows the user to better formulate the multiple objective functions for large optimization runs. Alternatively in our researches utilizing RSO e.g. [3], [4], [25], [27], [28], which advocate learning for optimization, the algorithm selection, adaptation and integration, are done in an automated way and the user is kept in the loop for subsequent refinements and final decision-making [10]. Here one of the crucial issue in MCDM is to critically analyzing a mass of tentative solutions related to materials and draping simulation, which is visually mined to extract useful information. Concerning solving the MCDM problems the DM is not distracted by technical details instead concentrates on using his expertise and informed choice among the large number of possibilities. As the whole process may be carried out in different design and design-making departments worth mentioning that the workflow may overlaps with a number of other fields of research such as enterprise decision management [26].

6. Software architecture of the reactive and interactive MCDM visualization environment

The proposed software is based on a three-tier model, independent from the optimization package [25], [28], which is an effective and flexible software architecture for integrating problem-solving and optimization schemes into the integrated engineering design processes and optimal design, modeling, and decision-making well suited for optimal design of textile composites [5] and [6]. The software is implemented a strong interface between the generic optimization algorithm and DM. While optimization systems produce different solutions, the DM is pursuing conflicting goals and tradeoff policies represented on the multi-dimensional graphs (see Fig. 4 and Fig. 5).

7. Discussion and conclusions

In this paper a novel environment for optimization, analytics and decision support in general engineering design problems is introduced. The utilized methodology is based on reactive search optimization procedure and its recently implemented software packages. The new set of powerful integrated data mining, modeling, visualization and learning tools via a handy procedure stretches beyond a decision-making task and attempts to discover
new optimal designs relating to decision variables and objectives, so that a deeper understanding of the underlying problem can be obtained.

Fig. 4. Mechanical modeling of draping process for a number of draping degrees.

Fig. 5. a) Parallel chart considering five optimization objectives simultaneously  b) The 7D visualization graph used for considering different products, materials and draping characteristics simultaneously.

Here along with presenting a study case, the interactive procedure is introduced which involves the DM in the optimization process helping to choose a single solution at the end. The method is well capable of handling the big data often associated with MCDM problems. Along with presenting the study case the aspects of data mining, modeling, and visualization the data related to material selection are considered. Further the utilization of the proposed software architectures for multiobjective optimization and decision-making, with a particular emphasis on supporting flexible visualization is discussed. The applicability of the software can be easily customized for different problems and usage contexts. Yet considering the abilities of visualization, the interesting patterns are automatically extracted from our raw data-set via data mining tools. Additionally the advanced visual analytical interfaces are involved to support the DM interactively. With utilizing the features such as parallel filters and clustering tasks, in the materials selection study case the managers can solve MOO problems as it amends previous approaches. The
utilization of a software architecture for MCDM, including the mechanical modeling of draping, with a particular emphasis on supporting flexible visualization is discussed. The applicability of software can be easily customized for different problems and usage contexts. The preliminary tests of the software environment in the concrete context of designing a multiple dome shapes have shown the effectiveness of the approach in rapidly reaching a design preferred by the decision-maker.

References


Two-sided guaranteed estimates of the cost functional for optimal control problems with elliptic state equations

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Abstract. In the paper, we discuss error estimation methods for optimal control problems with distributed control functions entering the right-hand side of the corresponding elliptic state equations. Our analysis is based on a posteriori error estimates of the functional type, which were derived in the last decade for many boundary value problems. They provide guaranteed two-sided bounds of approximation errors for any conforming approximation. If they are applied to approximate solutions of state equations, then we obtain new variational formulations of optimal control problems and guaranteed bounds of the cost functional. Moreover, for problems with linear state equations this procedure leads to guaranteed and computable error estimates for the state and control functions.

Keywords: a posteriori error estimates, elliptic boundary value problems, optimal control problems, guaranteed error bounds.

1. Introduction

Mathematical foundations of optimal control and various numerical methods are well developed (see, e.g., [8] and [1]). In the majority of cases, optimal control problems can be considered within the framework of in the following abstract form. Consider a certain cost functional $J(\eta, v): \Xi \times U \to \mathbb{R}$ where $\Xi$ and $U$ are reflexive Banach spaces (associated with state and control functions, respectively). The goal is to find $u \in K_u = \{ u \in U \}$ such that

$$J(\eta, u) = \inf_{v \in K_u} J(\eta, v),$$

where $K_u$ is a closed set of admissible control functions and $\eta \in \Xi$ solves the problem

$$\mathcal{A}(\eta, v) = 0.$$  

Here, $\mathcal{A}$ is a certain (linear or nonlinear) differential operator. It is assumed that the problem (2) is well posed and the cost functional $J$ is bounded from below and continuous with respect to both variables.

We consider a subclass of optimal control problems, in which the control function $v$ enters the source term of the equation, i.e., problems of the type

$$\mathcal{A}(\eta) = v + f,$$  

where $f$ is a given function in the image space of the operator $\mathcal{A}$. In the simplest case, (3) is generated by the linear boundary value problem: Find $\eta \in V_0(\Omega) := \{ \eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma \}$ such that

$$\int_{\Omega} A\nabla \eta \cdot \nabla w \, dx = \int_{\Omega} (v + f) w \, dx, \quad \forall w \in V_0(\Omega),$$

where $A$ is a given operator.
where $\Omega$ is a bounded connected domain in $\mathbb{R}^d$ with Lipshitz boundary $\Gamma$, $A$ is a symmetric positive definite matrix such that $\nu_1 |\xi|^2 \leq A \xi \cdot \xi \leq \nu_2 |\xi|^2$, and the functions $v$ and $f$ belong to $L^2(\Omega)$. Let $\sigma(\eta) := AV \eta$ be the flux associated with $\eta$. For this problem, integral type cost functionals are often used $(\alpha > 0)$

$$\begin{align*}
J_1(\eta, v) &= \frac{1}{2} \left\| \sigma(\eta) - \sigma^d \right\|^2_{A^{-1}} + \frac{\alpha}{2} \left\| v - u^d \right\|^2 \\
J_2(\eta, v) &= \frac{1}{2} \left\| \eta - \eta^d \right\|^2 + \frac{\alpha}{2} \left\| v - u^d \right\|^2.
\end{align*}$$

(5)

Here $\| \|$ denotes the norm of $L^2$ (since no confusion may arise we use the same notation for scalar and vector valued functions), $u^d, \eta^d$, and $\sigma^d \in L^2(\Omega, \mathbb{R}^d)$ are given functions representing the desired flux and the control function, respectively. In this case, $Z = V_0(\Omega)$ and $U = L^2(\Omega)$. In more complicated cases, $\mathcal{A}$ can be represented by a nonlinear boundary value problem (e.g., by a variational inequality). The set of admissible control functions may include constrains, i.e., $K_U := \{ v \in L^2(\Omega) \mid v \leq v_\oplus \text{ a.e. in } \Omega \}$, $v_\oplus \in L^\infty(\Omega)$. It is well known that under the above assumptions Problems 1 and 2 have unique solutions (e.g., see [8]). Approximation methods, a priori and a posteriori estimates, and adaptive numerical methods were intensively studied in the last decade (see, e.g., [2], [3], [4], [6], [7], [11] and other publications cited in these papers).

Our goal is to deduce fully computable error bounds for approximate solutions of optimal control problems.

### 2. Two-sided bounds of the cost functional

The key mathematical tools used to establish two-sided and guaranteed bounds of cost functionals comes from the theory of functional type a posteriori error estimates, which provides a guaranteed bound of the difference between the exact solution of a boundary value problem and any conforming approximation from the corresponding energy class (see [14] – [18]). In terms of (1) – (2) these estimates reads as follows:

$$M^-(\eta, v, D) \leq \| \eta - \eta \|_{\Sigma} \leq M^+(\eta, v, D).$$

(6)

Here $M^-(\eta, v, D)$ and $M^+$ are explicitly computable functionals. They depend on the control function $v$, the corresponding approximate solution $\eta$, and other explicitly known data $D$. In the last decade, estimates (6) has been derived for many problems generated by elliptic and parabolic differential equations. In [10], the reader will find a consequent exposition of the corresponding numerical methods and algorithms. For example, consider the problem

$$\text{div } A \nabla \eta_g + g = 0 \quad \text{in } \Omega, \quad \eta_g = \mu \quad \text{on } \Gamma, \quad g \in L^2(\Omega).$$

(7)

Let $\eta \in H^1_0(\Omega) + \mu$ be an approximation of $\eta_g$, which satisfies the Dirichlet boundary condition. Then,

$$\| \nabla(\eta - \eta_g) \|_A \leq \| A \nabla \eta - \tau \|_{A^{-1}} + C_{F \Omega} \| \text{div } \tau + f \|,$$

(8)

where

$$\| \tau \|^2_A := \int_\Omega A \tau \cdot \tau \, dx, \quad \| \tau \|^2_{A^{-1}} := \int_\Omega A^{-1} \tau \cdot \tau \, dx, \quad \forall \tau \in L^2(\Omega, \mathbb{R}^d),$$

and $C_{F \Omega}$ is the constant in the Friedrichs inequality (for functions vanishing at the boundary).
Estimates of the type (aposteriori PDE) can be applied to approximate solutions of state equations. For example, if (8) is applied to (4), then we have
\[
\|\mathcal{V}(\eta - \eta_\delta)\|_A \leq \|A\nabla \eta - \tau\|_{A^{-1}} + C_{F\Omega}\|\operatorname{div} \tau + v + f\|.
\] (9)

Majorants and minorants of the functional type derived for many linear and also nonlinear problems possess the following important properties: they are continuous with respect to both variables and for any \(v \in U \) and \(\eta \in V_0\)
\[
M^-(\eta, v, \mathcal{D}) \text{ and } M^+(\eta, v, \mathcal{D}) \text{ are nonnegative functionals};
\]
\[
M^-(\eta_v, v, \mathcal{D}) = M^+(\eta_v, v, \mathcal{D}) = 0.
\] (10) (11)

Now, we can discuss the general result associated with the setting (1) – (2). Assume that the cost functional satisfies the following condition: for any \(v \in K_U\)
\[
f(\eta, v) - \Psi(\|\vartheta\|_\Xi) \leq f(\eta + \vartheta, v) \leq f(\eta, v) + \Phi(\|\vartheta\|_\Xi),
\] (12)

where \(\Phi \) and \(\Psi \) are some known (continuous) functions vanishing at zero. We note that (12) can be viewed as a continuity condition with respect to the state function. In the majority of cases, this condition holds.

**Theorem 1.** Let (12) and (6) hold. Then
\[
\sup_{\eta \in \Xi} \inf_{v \in K_U} J^- \leq \inf_{\eta \in \Xi} \inf_{v \in K_U} J^+ \]
\[
\text{where } J^+(\eta, v) := f(\eta, v) + \Phi\left(M^+(\eta, v, \mathcal{D})\right) \text{ and } J^- : \eta, v) := J(\eta, v) - \Psi(M^-(\eta, v, \mathcal{D})).
\] (13)

Consider the problem (3) and (5) as a particular case. Let \(v \in K_U\) be an approximation of the exact control \(u\). By \(\eta_v\) we denote the corresponding exact solution of the state equation. In general, \(\eta_v\) is unknown and we use a certain approximation \(\eta \in V_0(\Omega)\) instead. It is easy to see that
\[
J(\eta_v, v) \leq \frac{1}{2} \left(\|\sigma(\eta) - \sigma^d\|_{A^{-1}} + \|\sigma(\eta_v) - \sigma(\eta)\|_{A^{-1}}\right)^2 + \frac{\alpha}{2} \|v - u^d\|^2.
\]

We apply (9) and find that
\[
\|\sigma(\eta_v) - \sigma(\eta)\|_{A^{-1}} = \|\nabla(\eta_v - \eta)\|_A \|\tau - \nabla\eta\|_{A^{-1}} + C_{F\Omega}\|\operatorname{div} \tau + v + f\|.
\]

Hence, we find that for any \(\eta \in V_0\) and \(v \in K_U\)
\[
J(\eta_v, v) \leq \frac{1}{2} \left(\|\sigma(\eta) - \sigma^d\|_{A^{-1}} + \|\tau - A\nabla\eta\|_{A^{-1}} + C\|\operatorname{div} \tau + v + f\|\right)^2 + \frac{\alpha}{2} \|v - u^d\|^2.
\]

Note the the right-hand side of this estimate is fully computable. Guaranteed lower bounds are derived by a different method (see [5]).

**Theorem 2.** (i) For any \(\eta \in V_0\), \(\tau \in H(\Omega, \operatorname{div})\) or \(\tau \in \bar{H}^N(\Omega, \operatorname{div})\), and positive \(\alpha\) and \(\beta\)
\[
\inf_{\eta \in V_0, \alpha, \beta > 0} \inf_{\tau \in H(\Omega, \operatorname{div})} J^+(\alpha, \beta, \tau, v),
\]
\[
J^+(\alpha, \beta, \tau, v) := j_{11}(\alpha; \eta) + j_{12}(\alpha, \beta; \eta, \tau) + j_{13}(\alpha, \beta; \tau, v).
\] (14)

where
\[ j_{11}(\alpha; \eta) := \frac{1 + \alpha}{2} \| \sigma(\eta) - \sigma^d \|_{A^{-1}}, \quad j_{12}(\alpha, \beta; \eta, \tau) := \frac{(1 + \alpha)(1 + \beta)}{2\alpha} \| \tau - A\nabla \eta \|_{A^{-1}}, \]

\[ j_{13}(\alpha, \beta; \tau, v) := \frac{c_{a\beta}}{2} \| \text{div} \tau + v + f \| + \frac{a}{2} \| v - u^d \|, \]

and \( c_{a\beta} = C \frac{2(1 + \alpha)(1 + \beta)}{a\beta}. \)

(ii) \( J_1(\eta, u) \geq J_1^-(\eta, w) \) for any \( \eta \) and \( w \) in \( V_0 \), where

\[ J_1^-(\eta, w) := G(\eta, w) + \frac{1}{2} \| \nabla(\eta - \eta^d) \|_{A}^2 + \int_{\Omega} \left( f(\eta - \eta^d) - A\nabla \eta \cdot \nabla(\eta - \eta^d) + \mathcal{H}(a, u^d, v^\oplus, w + \eta - \eta^d) \right) \, dx, \]

\[ \mathcal{H}(a, u^d, v^\oplus, g) := \begin{cases} u^d g - \frac{1}{2a} g^2 & \text{if } \bar{v} = u^d - \frac{g}{a} \leq v^\oplus, \\ v^\oplus g + \frac{a}{2} (v^\oplus - u^d)^2 & \text{if } \bar{v} > v^\oplus. \end{cases} \]

3. Estimates for the state and control functions

Guaranteed error majorants can be also derived for the error measured in terms of the combined norm

\[ \| u - v \| := \frac{1}{2} \| \nabla(\eta_u - \eta_v) \|_{A}^2 + \frac{a}{2} \| u - v \|. \]

**Theorem 3.** For any \( v \in L^2(\Omega) \),

\[ \| v - u \| \leq M^+(\alpha, \beta, \eta, \tau, w, v), \]

where \( \tau \in H(\Omega, \text{div}), w \in V_0(\Omega), \) and \( M^+(\alpha, \beta, \eta, \tau, w, v) := J_{1,\alpha,\beta}^+(\eta, \tau, v) - J_1^- (\eta, w) \geq 0. \)

**References**


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Abstract. When designing polymer nano-composites for electrical applications, the capacity to predict the electrical conductivity of the material as function of filler fraction is important. Finite element analysis (FEM) can be used to estimate the composite conductivity assuming that the properties of the components are known in detail and that sufficient computer capacity is available. In this study, FEM conductivity simulations for composites containing spherical inorganic nanoparticles in a polymer matrix were conducted, using different geometries. The simulations were performed both with and without interfacial layers between the particle surfaces and the surrounding polymer matrix. It was concluded that interfaces have a major impact on the simulation results and that the choice of geometrical model influences the results heavily, especially at high volume fractions of fillers.

Keywords: polymer composite, meso-scale modelling, conductivity, FEM.

1. Introduction

Polymer materials are extensively used in electrical insulation applications. Important dielectric properties to control are the relative dielectric constant, loss factor (tan delta), breakdown strength and resistivity. These parameters are influenced by the molecular structure of the polymer chains, the degree of crystallinity and the addition of different insulating or semi-conductive filler particles. The type of dispersion of the added filler particles is also of great importance. Currently this optimization process is based on an empirical approach during formulation and compounding of the polymer-based insulation materials. It is thus highly desirable to be able to predict the dielectric properties of various polymer-based insulation materials using material simulations. For example, assuming that the material properties of all constituents are known in advance, estimates of composite conductivity as function of filler content and geometrical arrangement can be done. Effective media finite element modelling of polymer composites with focus on electrical properties has been conducted in a number of previous studies. For instance, Nilsson et al. studied the permittivity of polymers filled with anisotropic fillers [1], Wang et al. studied the conductivity for composites with elongated fillers [2], Wenkatesulu et al. studied nano-composites with spherical fillers including an spherical interaction zone between the nanofiller and polymer matrix [3], Cheng examined two-phase disordered composites [4] while statistically isotropic composites was studied by Myroshnychenko et al. [5]. Filler percolation phenomena in polymers have been studied for instance by Kirkpatric [6], Webman [7] and Mårtensson [8]. Theoretical models for spherical nano-particles surrounded by interaction zones have been derived by Tanaka [9], [10]. The aim of this study is to examine the pros and cons of different strategies for modelling polymer composites on the finite element level. Special attention is drawn to the effect of including an interaction zone around the particles.
2. Models and methods

In the study, equation (1) was solved on a box-shaped domain with FEM using Comsol Multiphysics 4.3a with Matlab. The composite conductivity was called $\sigma$ [A/Vm] while the filler-, the matrix- and the interface conductivities were called $\sigma_1, \sigma_2$ and $\sigma_3$ respectively. The voltage was denoted $V$ [V].

$$\nabla \cdot (\sigma \nabla V) = 0. \quad (1)$$

The lowermost face of the box was grounded ($V = 0$), the uppermost face was connected to a voltage source ($V = V_0$) and the remaining four faces were given peridical boundary conditions ($V_a = V_b$). The composite conductivity in the vertical $z$-direction was calculated by first integrating the current density in the $z$-direction and then dividing the result with the applied voltage and the area of the lowermost face. All the simulated composites consisted of a polymer matrix containing spherical fillers with or without surrounding interfaces. The geometries of the composites were constructed in three different ways: (1) with the “random oriented object” (ROO) model where the particles were positioned completely randomly inside the domain, (2) with the “random oriented object on a grid” (ROOG) model where the spheres were positioned randomly on an underlying hexagonal grid and (3) with the “smallest repeating box” (SRB) model where the particles filled all positions of a hexagonal grid. The geometries were all periodical. Examples of the three kinds of geometries are shown in Fig. 1a) - Fig. 1c) where the colour is proportional to the electric field ranging from $V_0$ (red) to 0 (blue). In these figures the relative particle conductivity is $\sigma_1/\sigma_2 = 1000$, the relative interface conductivity is $\sigma_3/\sigma_2 = 100$, the volume fraction ($\phi_1$) of particles is approx 30 vol% and the thickness of the interlayer is 10% of the particle radius.
c) Fig. 1. Sample geometries with a) the ROO model b) the ROOG model and c) the SRB model. The color is proportional to the electric field. The volume fraction ($\phi_1$) of particles is approx 30 vol%. It can be noted that the probability for continuous perculated paths seem to be largest with the ROOG model and smallest with the SRB model.

3. Results and discussion

In Fig. 2a), homogenous monodisperse composites without boundary layers were studied with the SRB model. The relative composite conductivity $\sigma_{rel} = \sigma/\sigma_2$ was plotted against the volume fraction of fillers ($\phi_1$). Six different $\sigma_1/\sigma_2$ fractions are used. The percolation threshold becomes steep and positioned slightly above $\phi_1 = 0.7$, where the particles approaches close contact. Since the distances between the particles are equal and maximized, the SRB model without an interlayer in practice gives the lower theoretical bound of the composite conductivity. In Fig. 2b), results for the corresponding system with thin interlayers are presented. The thickness of the layer was chosen as 10% of the radius of the sphere, which is the same as the experimentally observed interlayer thickness for epoxy-hollow glass sphere composites examined in Nilsson [1]. The relative interface conductivity $\sigma_3/\sigma_2$ was modelled as $\sigma_3/\sigma_2 = (\alpha \sigma_1 + (1 - \alpha) \sigma_2)/\sigma_2$ with the parameter $0 \leq \alpha \leq 1$ chosen to $\alpha = 0.1$. By including this thin interface, the position of the percolation threshold was lowered down to $\phi_1 = 0.55$ and the composite conductivity above the percolation threshold exhibited an exponential relationship in accordance to predictions with percolation theory [6]. Detailed information about the interlayer properties are thus needed for accurate predictions without adjustable parameters, which would otherwise be tempting to use.

In Fig. 3a) the three different ways of constructing geometries were compared for composites without interlayers. The relative filler conductivity was $\sigma_1/\sigma_2 = 1000$ and the relative interface conductivity $\sigma_3/\sigma_2$ was chosen as $\sigma_3/\sigma_2 = 100.9$, i.e. $\sigma_3/\sigma_2 = (\alpha \sigma_1 + (1 - \alpha) \sigma_2)/\sigma_2$ with $\alpha = 0.1$. Ten iterations of each setting was performed. The data of the ROO model became positioned close to the results of the SRB model, which formed a theoretical lower bound of the composite conductivity. The ROOG systems became percolated at much lower degree of filling and gave much higher conductivity values.
Fig. 2. SRB model results a) without an interlayer and b) with an interlayer. Filler volume fraction was plotted against relative composite conductivity for 6 different $\sigma_1/\sigma_2$ fractions.

In Fig. 3b) corresponding results were plotted for systems including interlayers with a thickness of 10% of the particle radius. At relatively high filler fractions (0.17 – 0.30), the composite conductivity of the ROOG model was increased by a factor 5 due to the inclusion of the thin interlayer while the conductivity of the ROO model was increased by a factor 2.5. The choice of geometrical representation as well as the presence of a boundary layer thus heavily influenced the composite properties.

Fig. 3. Comparisons between ROO, ROOG and SRB a) without an interlayer and b) with an interlayer. Filler volume fraction was plotted against relative composite conductivity $\sigma_1/\sigma_2 = 1000$ and $\sigma_3/\sigma_2 = 100.9$.

4. Conclusions

Finite element calculations for modelling the relative composite conductivity as function of filler fraction have been conducted for composites with spherical nano-particles. Three models for constructing the geometries have been proposed: “The Smallest Repeating
Box” (SRB) model, the “Random Oriented Object” (ROO) model and the “Random Oriented Object on a Grid” (ROOG) model. In addition, the effect of including a boundary phase surrounding the particles has been studied. The SRB model without interfaces is a lower bound for composite conductivity calculations when used on a hexagonal grid. Due to the underlying grid which increases the possibility of close contacts between particles, the ROOG model gives much higher conductivity values than the ROO model, a fact to be aware of when constructing on-lattice conductivity models, which is typically the case in electrical network models. On the other hand, when designing FEM models with randomly positioned objects like in the ROO model, the effect of local aggregations must be included, otherwise the calculated conductivity will become too low. Furthermore, since the effect of including a thin interlayer is significant, molecular simulations of the particle-matrix interface properties would clearly increase the FEM prediction power.

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The variables doubling in ordinary differential equations and equations systems

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Abstract. The dynamic systems trajectories space is weakly dense immersed in the stream of extremals. The doubling algorithm implemented (designed) the extremals flow immersion to the trajectories bunch is considered. The system energy (Hamiltonian) for linear systems with constant coefficients \( \dot{x} = Ax + B \), where \( A \) and \( B \) are numerical matrices (i.e. \( A \) is Jordan matrix) is constructed.

Keywords: Hamiltonian, linear system, Jordan matrix, doubling variables.

1. Introduction

The method of additive doubling variables, leading any stand-alone system to a canonical (Hamiltonian) system was announced in [1] (see lecture 30). Simple facts are installed in [2]:

1° there is a bunch of extremals \( (x,y(t)) \), such that one is locally \( C^1 \)-close to the trajectory of the original equation (the equations system);
2° every autonomous system \( \dot{x} = Ax \) generates a Hamiltonian
\[
E(x, y) = \int_0^x f(x + z)dz \Rightarrow E(x, 0) = 0 \quad \text{and} \quad \dot{x} = f(x + y), \dot{y} = f(x) - f(x - y);
\]
3° let \( \|y\| \ll 1 \), then the canonical system coincides with the equations system in variations.

2. General case

Let us consider linear system with constant coefficients
\[
\dot{x} = Ax + B, \quad x, B \in \mathbb{E}^m, \quad A: \mathbb{E}^m \to \mathbb{E}^m, \quad x(0) - x_0 = 0.
\]
The system is canonic if and else \( \text{Sp}(A) = 0 \). This is generally not true.
Let us assume there is the procedure of full additive doubling variables. In this case, the \( y \)-gradient of the Hamiltonian is determined as follows:
\[
\frac{\partial E}{\partial y} = A(x + y) + B,
\]
\[
E(x, y) = -\frac{1}{2} \sum_{i \leq m} a_{ii} y_i + \sum_{i,j \leq m} y_i a_{ij} x_j + \sum_{i,j \leq m} y_i a_{ij} y_j = \frac{1}{2} \|y\| + \langle y, Ax \rangle + \langle y, Ay \rangle
\]
Then the doubled system of equations has the form:
the block matrix $\alpha$ determined as follows

$$\alpha = \begin{pmatrix} A & A' - \text{diag} A \\ 0 & -A' \end{pmatrix}$$

and vector $\beta = \text{col}(B, 0, ..., 0)$.

Then $\text{sp}(\alpha) = 0$ and the doubling system really is canonic (Hamiltonian system) in whole and for each pair component $x_i, y_i$. The role of the energy system plays $E(x, y)$.

Let us assume $y(0) = y_0$. It is clear that if $y_0 \neq 0$ then $y(t) \neq 0$ and the solution of the doubling system coincides with the extremal in the neighborhood $U(y_0)$: the trajectory $O_x(y)$ of a doubling system is a projection of the beam extremal on the original system trajectories set. If $y_0 \to 0$ then $x(y, t)$ uniformly converges to the solution of the original system $x(t)$ on $y \in U(y_0)$.

For instance let us assume $y$ is small. Then the doubling system as pulse equation is equation in variations and degenerate Hamiltonian $E(x, y) = yf(x)$ is not associated with any extreme condition. Therefore additive doubling implements the shift of a set of initial trajectories $O_x$ on the set of extremals $x(t, y)$.

A procedure of partial additive doubling of variables leads to the system

$$x_i = a_{ii}(x_i + y_i) + \sum_{j \neq i} a_{ij}x_j + b_i, \quad y_i = -a_iy_i, \quad i = 1, m,$$

such that one has Hamiltonian

$$E(x, y) = \langle B, y \rangle + \langle y, Ax \rangle + \frac{\|y\|}{2\|A\|}.$$ 

Then doubling system matrix $\alpha$ is equal

$$\begin{pmatrix} A & \text{diag} A \\ 0 & -\text{diag} A \end{pmatrix}$$

and $\text{Sp}(\alpha) = 0$. The statement about immersion trajectories doubling system in an extremals bunch is true.

3. Jordan case

Let us assume $p$ is matrix, such that $\dim p = m$ and $\det p \neq 0, A = p^{-1}jp$ ($j$ is the Jordan matrix). This transform is only up to the matrices equivalence. Then original system defined by the form

$$\dot{x} = p^{-1}jp\dot{x} + b.$$

Let us define $X \equiv px$ and $B \equiv pb$. Then original system defined by the form

$$\dot{X} = jX + B.$$

It is clear that there are two cases.

The scalar case is trivial. It is clear that full and partial additive doubling of variables is an identity and double system matrix $\alpha$ is equal

$$(A \begin{pmatrix} 0 \\ 0 \end{pmatrix}).$$

If there are attached vector, the initial system has the form

$$\dot{X}_0 = \lambda X_0 + B_0, \quad \dot{X}_i = \lambda X_i + X_{i+1} + B_i, \quad i = 1, m - 1.$$ 

Full additive doubling is generated by a Hamiltonian $E(X, Y)$ such that
\[ \lambda(X_0 + Y_0) + B_0 = \frac{\partial E}{\partial Y_0}, \quad \lambda(X_i + Y_i) + X_{i+1} + B_i = \frac{\partial E}{\partial Y_i}, \quad i = 1, m - 1, \]

and expression for the Hamiltonian is obtained

\[ E(X, Y) = \frac{\lambda}{2} \sum_{i \leq m} Y_i^2 + \lambda \sum_{i \leq m} X_i Y_i + \sum_{i \leq m} (X_{i+1} Y_i + Y_{i+1} Y_i) + \sum_{i \leq m} B_i Y_i \]

or

\[ E(X, Y) = \frac{\lambda}{2} \|Y\|^2 + \lambda \langle X, Y \rangle + \langle \tau(X + Y), Y \rangle + \langle B, Y \rangle \]

where \( \tau \) is \( m \)-nilpotent shift operator, such that \( \tau^m = 0 \) and \( \tau^{m-1} \neq 0 \).

Therefore the doubling system has the form

\[
\begin{align*}
\dot{X}_0 &= \lambda(X_0 + Y_0) + B_0, \\
\dot{X}_i &= \lambda(X_i + Y_i) + X_{i+1} + B_i, \quad i = 1, m - 1 \\
\dot{Y}_0 &= -\lambda Y_0, \\
\dot{Y}_i &= -\lambda Y_i - Y_{i+1}, \quad i = 2, m.
\end{align*}
\]

The matrix of doubling system is

\[
\begin{pmatrix}
  j & j' - \text{diag } j \\
  0 & -j'
\end{pmatrix}.
\]

The energy of doubling system at incomplete enhancer is determined as following

\[
\frac{\partial E}{\partial Y_0} = \lambda(X_0 + Y_0) + B_0, \quad \frac{\partial E}{\partial Y_i} = \lambda(X_i + Y_i) + X_{i+1} + B_i, \quad i = 1, m - 1.
\]

Then

\[ E(X, Y) = \frac{\lambda}{2} \|Y\|^2 + \lambda \langle X, Y \rangle + \langle \tau X, Y \rangle + \langle B, Y \rangle \]

and doubling system is still block

\[
\begin{pmatrix}
  X \\
  Y
\end{pmatrix} =
\begin{pmatrix}
  j & \text{diag } j \\
  0 & -j'
\end{pmatrix}.
\]

References


Using orthogonality-relation for the separation of Lamb modes at a plate edge

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Abstract. Lamb modes are widely used for non-destructive evaluation of plate-like structures and simple interpretation procedures for the analysis of the monitored structures are needed. In this study we apply the orthogonality relation-based method for post-processing Finite Element (FE) predictions in order to separate Lamb modes at a plate edge. The reflected wave field from the free edge is assumed to be a superposition of all the eigenmodes of an infinite plate. The eigenmode coefficients of the reflected wave field are determined by adapting the orthogonality-based method that was used to determine the reflection coefficients of Lamb modes at a plate edge. Overlapping wavepackets of Lamb modes at a plate edge are simulated by using the FE model of the incident S₀ mode in a plate with a crack. Time-domain signals of propagating, non-propagating and complex modes are extracted.

Keywords: Lamb modes, orthogonality, mode separation.

1. Introduction

Lamb modes are widely used for non-destructive evaluation of plate-like structures. Among the various challenges, the separation of modes is needed for the development of proper interpretation procedure to analyze the response of the monitored structures [1]. In general it is not possible to avoid multimodality in Lamb wave testing. Even if the incident wave is a pure Lamb mode, the interaction of a wave with a defect or structural feature can result in a complicated multimode signal, since there may exist at least two propagating modes in a plate at any chosen testing frequency.

Lamb modes can be separated from a signal by applying a classical two-dimensional spatial Fourier transform technique that uses time records from a series of equally spaced points along a plate [2]. However, this technique requires long paths of the accessible plate surface to be monitored, while in orthogonality-based method the number of measurement points of the through-thickness wave field can be significantly reduced [3]. One the other hand, in the proposed method the measurement of all displacement and stress field components is required which in practice can be very complicated. At the plate edge the orthogonality-relation is simplified as the stresses equal to zero. Therefore only in-plain and out-of-plain displacement components have to be measured at a plate edge.

The orthogonality of the Lamb modes was shown already a long time ago [4] and this property has helped to solve several wave propagation and scattering problems in structures. The idea of using orthogonality to extract individual Lamb modes from the scattered wave fields is not new. L. Moreau et al. [3] used the orthogonality relation to calculate the reflection and transmission coefficient of isolated modes in case of a pure Lamb mode incident on a notch-like defect. In their paper [5] they also showed that the
The proposed technique can be extended on three-dimensional guided wave scattering problems in plates. The use of modal decomposition method with the orthogonality relation has allowed to solve the Lamb wave interaction with a plate edge [6] and delaminated plate [7]. Gunawan and Hirose [8] derived a generalized orthogonality relationship for the Lamb modes of oblique scattering on the free edge of a plate. They used it to develop a mode decomposition technique for an elastodynamic field and semi-analytically obtained the reflection coefficients for the oblique incidence problem.

In this paper we present the orthogonality relation-based method for post-processing Finite Element (FE) predictions in order to separate Lamb modes at a plate edge in a plane strain condition. The reflected wave field from the free edge is assumed to be a superposition of all the eigenmodes of an infinite plate. The eigenmode coefficients of the reflected wave field are determined by adapting the orthogonality-based method that was used to determine the reflection coefficients of Lamb modes at a plate edge [6]. Overlapping wavepackets of Lamb modes at a plate edge are simulated by using the FE model of the incident S0 mode in a plate with a crack. Time-domain signals of propagating and non-propagating modes are extracted.

2. Orthogonality-relation of Lamb modes at a plate edge

Fig. 1 shows the two-dimensional Lamb mode propagating towards the edge of a semi-infinite plate along the x-direction. The plate medium is considered to be isotropic, homogeneous and elastic; plane strain conditions are considered.

![Fig. 1. Geometry of the problem.](image)

The displacements and stresses of each Lamb mode with order $n$ are expressed in the vector form:

$$\mathbf{u}_n = \begin{pmatrix} u_x \\ u_y \end{pmatrix}_n e^{i (\omega t - k_n x)},$$

$$\mathbf{\sigma}_n = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \end{pmatrix}_n e^{i (\omega t - k_n x)},$$

where $t$ is time, $\omega$ is angular frequency and $k_n$ is the complex wave number of mode $n$.

The reflected wave field can be written as a modal series of Lamb eigenmodes which must satisfy stress-free boundary conditions on the free edge $x = 0$:

$$\begin{pmatrix} S_{xx} \\ S_{xy} \end{pmatrix} = \sum_m r_m \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \end{pmatrix}_m = 0, \quad \begin{pmatrix} U_x \\ U_y \end{pmatrix} = \sum_m r_m \begin{pmatrix} u_x \\ u_y \end{pmatrix}_m,$$

where $r_m$ is the complex reflection amplitude of mode $m$.

The general orthogonality relation [3], [7], which involves a scalar product between the displacement and stress distributions of two modes $m$ and $n$ is considered at a given position along the plate:
where \( \delta_{mn} \) is the Kronecker delta symbol.

Applying the orthogonality condition to the reflected field on the free edge, complex reflection amplitudes of any mode \( n \) can be calculated knowing only the edge displacement field \( (U_x; U_y) \)

\[
\int_0^h \left[ (\sigma_{xy})_n (u_y)_m - (\sigma_{xx})_n (u_x)_m \right] dy = a(n) \delta_{mn},
\]

and

\[
a(n) = 2 \int_0^h r_n \left[ (\sigma_{xy})_n (u_y)_n - (\sigma_{xx})_n (u_x)_n \right] dy
\]

\[
r_n = a(n) / 2 \int_0^h \left[ (\sigma_{xy})_n (u_y)_n - (\sigma_{xx})_n (u_x)_n \right] dy.
\]

3. Post-processing procedure for FE results

Fig. 2 shows the FE model for Lamb modes scattered by a crack and reflected at a plate edge. Wave propagation was simulated by using finite element modelling software Ansys [9]. A pure Lamb mode is generated on the left edge which propagates along the plate and interacts with a crack in a plate and reaches to the plate edge. The interaction phenomenon causes the scattering of Lamb modes, reflected from and transmitted past the crack. Multiple reflections can take place as the crack is rather close to the plate edge. The total acoustic field in the guide can therefore be very complicated since it results from the superposition of the incident and all the diffracted modes: a series of propagating modes, plus an infinite number of non-propagating or attenuated.

![Fig. 2. FE model of \( S_0 \) mode interacting with a crack and plate edge.](image)

The measured time-domain signals at the plate edge are transformed into frequency-domain. As the excitation signal is chosen to be a Hanning-windowed toneburst, the extraction procedure must be performed over a range of frequencies. At each frequency step the through-thickness displacements and stresses of the separatable mode are calculated and amplitudes are predicted at the plate edge by using the expression (7).
Finally, the spectrum of the extracted mode is transformed back into time-domain by using inverse Fourier’ transform.

4. Results and discussion

Finite element simulations were performed for a 4 mm thick aluminium plate where the scattering of $S_0$ mode by a 2 mm deep crack was studied. The incident mode is a 10-cycle Hanning-windowed toneburst with a center frequency 250 kHz. Fig. 3 shows the simulated time domain signal of $u_x$ displacement component measured at the upper corner of the plate right edge. The amplitude is normalized by the peak amplitude of the incident wave packet. The signal is complicated, it is composed of multiple wavepackets of $S_0$ plus mode converted $A_0$ mode at a crack and also a number of nonpropagating modes which all can overlap due to the closeness of the crack to the plate edge.

![Fig. 3. $u_x$ displacement of the plate edge in case of $S_0$ mode incident simulated by FE modelling.](image)

For the separation of the modes the displacements were monitored along the plate edge in 9 equally spaced points and the orthogonality condition was applied. Fig. 4a), b), c), d), e) shows the time domain signal of normalized $u_x$ displacement component of the separated $S_0+$, $A_0+$, $S_0-$, $A_0-$ and first order non-propagating mode $C_1 +$ of the symmetric type. "+" denotes the mode propagating in positive direction an “−“ in negative direction, respectively.

![Fig. 4. a) $u_x$ displacement of the separated $S_0+$ mode.](image)
It can be seen that the wave packets are clearly separated. The first arrival is $S_0$ mode which is followed by the slower mode-converted $A_0$ mode generated at the crack. All the propagating modes are reflected without mode-conversion at the plate edge as the
amplitude of back-propagating mode remains the same as forward-propagating modes. Non-propagating modes generated at the plate edge have small amplitudes at the frequency range used in this study.

In future work it is important to investigate the errors that might occur in the extraction procedure due to the deviations in wave propagation parameters in experimental measurements.

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A method for description and optimization the structure of multi-level selection procedure

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Abstract. A method of description and optimal design of the structure of multi-level selection procedure is presented. The set of feasible structures for such class of systems is defined. The representation of this set is constructed in terms of the graph theory. For the reduced statement two types of variable parameters are defined: for the level size and for the relations of adjacent levels. The choice of variable parameters guarantees the discrete-convexity of objective function. A class of iteration methods for solving the discrete-convex programming problem is derived. The method based on the extension of discrete-convex function to the convex function and on extension of discrete-convex programming problem to the convex programming problem. On each step of the iteration the calculation of the value of objective function is required only on some vertices of unit cube.

Keywords: multi-level selection and processing environment, discrete-convex function, convex extension, convex programming, discrete-convex programming.

1. Introduction

Large-scale selections problems can be decomposed in many different ways [2]. The current approach for describing and optimizing the structure of hierarchical systems is based on a multi-level partitioning of given finite set in which the qualities of the system may depend on the partitioning. Examples of problems of this class are aggregation problems, structuring of decision-making systems, database structuring, the problems of multiple centralization or decentralization, multi-level selection problems, multi-level tournament systems [1], multi-level distribution systems, different clustering problems. The advantage of the considered approach is that this choice of variables enables to extend the structure optimization problem to the convex programming problem. A finite steps algorithm converging to the global solution of this problem is presented.

2. Feasible set of multi-level selection systems

Suppose $m_i \times m_{i-1}$ matrix $Y_i = (y_{jr}^i)$ is an adjacent matrix of levels $i$ and $i-1$ ($i = 1, ..., s$) where

$$y_{jr}^i = \begin{cases} 1, & j-th element on level i connected with r-th element on level i - 1, \\ 0, & otherwise. \end{cases}$$

Suppose $m_0$ is the number of 0-level elements (level of object, level of non-ordered set). Consider all $s$-levels hierarchies, where nodes on level $i$ are selected from the given nonempty and disjoint sets and all selected nodes are connected with selected nodes on adjacent levels. All oriented trees of this kind form the feasible set of hierarchies [4]. The illustration of this formalism is given in Fig. 1.
Theorem 1. All hierarchies with adjacent matrixes \( \{Y_1, ..., Y_s\} \) from the described set of hierarchies satisfy the condition \( Y_s \cdot ... \cdot Y_1 = (1, ..., 1) \).

The assertion of this theorem is determined directly [5].

Fig. 1. Feasible set of structures.

Fig. 2. The creation and annihilation of levels.

The illustration of multiplication of adjacency matrices is given on Fig. 2. To the multiplication of adjacent matrices correspond the annihilation of levels. To the presentation an adjacent matrix as a product of two adjacent matrices correspond the creation a new level.

To the sequence of adjacent matrices \( \{Y_1, Y_2, Y_3, Y_4\} \) corresponds the hierarchy where the arcs are described with continuous lines. To the sequence of adjacent matrices \( \{Y_1, Y_3 \cdot Y_2, Y_4\} \) correspond the hierarchy where the arcs between the first and the second levels are described with dash lines and other arcs are described with continuous lines.
3. The statement of general problem of structure optimization

The general optimization problem is stated as a problem of selecting the feasible structure that corresponds to the minimum of total loss given in the separable-additive form:

$$\min_{y_1, \ldots, y_s} \left\{ \sum_{i=1}^{s} \sum_{j=1}^{m_i} h_{ij} \left( \sum_{r=1}^{m_{i-1}} d_{jr}^{i} y_{jr}^{i} \right) \right\} Y_s \cdot \cdots \cdot Y_1 = (1, \ldots, 1). \quad (1)$$

Here $h_{ij}(\cdot)$ is an increasing loss function of $j$-th element on $i$-th level and $d_{jr}^{i}$ is the element of $m_i \times m_{i-1}$ matrix $D_i$ for the cost of connection between the $i$-th and $(i - 1)$-th level. The meaning of functions $h_{ij}(k)$ depends on the type of the particular system.

4. The reduced problem of structure optimization

Now an important special case is considered where the connection cost between the adjacent levels is the property of the supreme level: each row of the connection cost matrices between the adjacent levels consists of equal elements. There is a possibility to change the variables and to represent the problem so that

$$d_{jr}^{i} = 1; \ i = 1, \ldots, s; \ j = 1, \ldots, m_i; \ r = 1, \ldots, m_{i-1}. \quad (2)$$

Now the total loss depends only on sums

$$\sum_{r=1}^{m_{i-1}} y_{jr}^{i} = k_{ij},$$

where $k_{ij}$ is the number of edges beginning in the $j$-th node on $i$-th level.

Recognize also that

$$\sum_{j=1}^{m_i} k_{ij} = p_{i-1}, \ i = 1, \ldots, s,$$

where $p_i$ is the number of nodes on $i$-th level. If to suppose additionally that $h_{i1}(k) \leq \ldots \leq h_{im_i}(k)$ for each integer $k$, the general problem (1) transforms into the two mutually dependent phases:

$$\min_{p_1, \ldots, p_{s-1}} \left\{ \sum_{i=1}^{s} g_i (p_{i-1}, p_i) \ \left| \ p_0 \geq p_1 \geq \cdots \geq p_s, p_s = 1 \right. \right\}, \quad (2)$$

where

$$g_i (p_{i-1}, p_i) = \min_{k_{i1}, \ldots, k_{ip_i}} \left\{ \sum_{j=1}^{n_i} h_{ij}(k_{ij}) \ \left| \ \sum_{j=1}^{n_i} k_{ij} = p_{i-1} \right. \right\}. \quad (3)$$

Free variables of the inner minimization (3) are used to describe the connections between the adjacent levels. Free variables of the outer minimization (2) are used for the representation of the number of elements at each level.
5. Convex extension of discrete-convex functions

This statement has some advantages from the point of view of the optimization technique. It is possible to adapt effective methods of the convex programming for solving outlined special cases. The function \( f: Z^n \to R \) is called discrete-convex \([3], [4]\) if for all

\[
z_i \in Z^n \quad (i = 1, \ldots, n + 1), \quad Z^n = Z \times \ldots \times Z, \quad Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\},
\]

\[
\lambda_i \geq 0, \quad \sum_{i=1}^{n+1} \lambda_i = 1; \quad \sum_{i=1}^{n+1} \lambda_i z_i \in Z^n \text{ holds } f \left( \sum_{i=1}^{n+1} \lambda_i z_i \right) \leq \sum_{i=1}^{n+1} \lambda_i f(z_i)
\]

The use of all \( n + 1 \) elements convex combinations follows from the well-known theorem of Caratheodory \([6]\).

The graph of a discrete-convex function is a part of the graph of a convex function.

The convex extension \( f_c \) of function \( f:X \to R \) \((X \subset R^n)\) is the majorant convex function \( f_c: \text{conv} \ X \to R \), where \( f_c(x) = f(x) \) if \( x \in X \).

**Theorem 2.** The function \( f:X \to R \) \((X \subset R^n)\) can be extended to convex function on \( \text{conv} \ X \) if \( f \) is discrete-convex on \( X \). The convex extension \( f_c \) of \( f \) is

\[
f_c(x) = \min \left\{ \sum_{i=1}^{n+1} \lambda_i f(x_i) \right\} \quad \left| \begin{array}{c} x = \sum_{i=1}^{n+1} \lambda_i x_i; \quad \sum_{i=1}^{n+1} \lambda_i = 1; \quad \lambda_i \geq 0 \quad (i = 1, \ldots, n + 1), \\
x_i \in X \quad (i = 1, \ldots, n + 1) \end{array} \right.
\]

over \( x_i, \lambda_i \quad (i = 1, \ldots, n + 1) \).

Assertion of this theorem is determined directly \([4]\).

**Theorem 3.** The convex extension \( f_c \) of \( f \) is

\[
f_c(x) = \begin{cases} \max \{(a,x) + b|(a,y) + b \leq f(y), y \in X\}, & \text{if } x \notin X, \\
f(x), & \text{if } x \in X. 
\end{cases}
\]

Assertion is determined directly.

From theorem 3, the convex extension is so called point-wise maximum over all linear functions not exceeding the given function.

From theorem 3, the convex extension of a discrete-convex function is a piecewise linear function.

From theorem 2 and/or 3, each discrete-convex function has a unique convex extension.

From theorem 2 and/or 3, the class of discrete-convex functions is the largest one to be extended to the convex functions.

**Theorem 4.** If \( h_{ij}(k) \quad (i = 1, \ldots, s; j = 1, \ldots, m_i) \) in (3) are discrete-convex functions then

\[
\sum_{i=1}^{s} g_i(p_{i-1}, p_i)
\]

in (2) is a discrete-convex function.

The proof of this theorem is not very complicated but needs a lot of secondary results and can be found in \([4]\).
Considered Theorem 2 and Theorem 4 enable to extend the objective function (2), (3) to the convex function.

6. Algorithm of local searching for the reduced problem of structure optimization

The particular choice of the variables (2), (3) enables to construct a class of methods for finding the global optimum. In this paper it is only declared that the objective function of such integer programming problem is a discrete-convex function.

Suppose additionally that the convex extension of the values of in vertices of the unit cube containing the . Then the expression in theorem 2 reduces to

\[
f_c(x) = \min_{y(\lambda_i)} \left\{ \sum_{i=1}^{2^n} \lambda_i f(y_i) \mid x = \sum_{i=1}^{2^n} \lambda_i y_i; \sum_{i=1}^{2^n} \lambda_i = 1; \lambda_i \geq 0 \right\},
\]

where are the coordinates of vertices of unit cube containing this . At least \(2^n - n - 1\) of \(\lambda_i\) are equal to zero.

Recall of (2)

\[
g_0(p_0, p_1, \ldots, p_{s-1}, 1) = \sum_{i=1}^{s} g_i(p_{i-1}, p_i)
\]

and denote

\[
p^{(s)}_k = (p^{(s)}_{k1}, \ldots, p^{(s)}_{ks-1}, 1) \quad (k = 0, 1, \ldots).
\]

Consider following finite-step algorithm:

\[
p^{(s)}_0 = (1, \ldots, 1) \text{ and } p^{(s)}_k = p^{(s)}_{k-1} + x^{(s)}_k(q, t), \quad (k = 1, 2, \ldots).
\]

It is assumed that

1) \(x^{(s)}_k(q, t)\) are lexicographically ordered by \((q, t)\):

\[
x^{(s)}_k(q, t) = \left( \begin{array}{c} s \\ \left( 0, \ldots, 0 \right)_{\frac{t}{\ell}}, \left( 1, \ldots, 1 \right)_{\frac{t}{q}}, 0, \ldots, 0 \end{array} \right),
\]

\((q = 1, \ldots, s - t - 1; \quad t = 0, \ldots, s - 2);\)

2) \(p_0 \geq p^{(s)}_{k1} \geq \ldots \geq p^{(s)}_{ks} = 1\)

3) \(x^{(s)}_k(q, t)\) is lexicographically the first that satisfies the condition

\[
g \left( p_0, p^{(s)}_{k1}, \ldots, p^{(s)}_{ks-1}, 1 \right) \leq g \left( p_0, p^{(s)}_{k-11}, \ldots, p^{(s)}_{k-1s-1}, 1 \right).
\]

Remark 1. Consider the vertices with integer valued coordinates of the \(s\)-dimensional unit cube where:

the nearest vertex to the \(s\)-dimensional zero-point is \(p^{(s)}_{k-1} = (p^{(s)}_{k-11}, \ldots, p^{(s)}_{k-1s-1}, 1);\)
other vertices $p_{k-1}^{(s)} + x_k^{(s)}(q, t)$ satisfy the condition (4).
The number of that kind of vertices (4) is $1/2 \cdot s(s - 1)$. The number of vertices (4), (5) is no more than $1/2 \cdot s(s - 1)$.
The condition (4) puts in order all vertices of described unit cube.

Remark 2. On the iteration step $k$ the value of goal function is computed on ordered vertices (4), (5) of unit cube until the first value satisfying (6) is found. If that kind of a value does not exist, one of the solutions of problem (2)–(3) has been found.

7. Conclusion

Many finite hierarchical structuring problems can be formulated mathematically as a multi-level partitioning procedure of a finite set of nonempty subsets. This partitioning procedure is considered as a hierarchy where to the subsets of partitioning correspond nodes of hierarchy and the relation of containing of subsets define the arcs of the hierarchy. The feasible set of structures is a set of hierarchies (oriented trees) corresponding to the full set of multi-level partitioning of given finite set.

Each tree from this set is represented by a sequence of Boolean matrices, where each of these matrices is an adjacency matrix of neighboring levels. To guarantee the feasibility of the representation, the sequence of Boolean matrices must satisfy some conditions – a set of linear and nonlinear equalities and inequalities.

The formalism described in this paper enables to state the reduced problem as a two-phase mutually dependent discrete optimization problem and construct some classes of solution methods. Variable parameters of the inner minimization problem are used for the description of connections between adjacent levels. Variable parameters of the outer minimization problem are used for the presentation of the number of elements on each level.

The two-phase statement of optimization problem guarantees the possibility to extend the objective function to the convex function and enables to construct algorithms for finding the global optimum. In this paper for finding the global optimum the method of local searching is constructed. On each step of iteration the calculation of the value of objective function is required only on some vertices of some kind of unit cube.

References

Effect of a multiscale factor on the Young’s modulus of a cylindrical shell

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Abstract. Effect of a multiscale factor on the Young’s modulus of a cylindrical shell is considered. Dependence of the Young’s modulus from quantity of molecular layers of a cylindrical shell is investigated. Research is based on the main equations of the theory of elasticity for a circular cylindrical shell made from a layered material.

Keywords: multiscale factor, Young’s modulus, circular cylindrical shell, layered material.

1. Introduction

The substance transition from macro- and micro- to the nanosizes involves high-quality changes in their physical, mechanical, physical and chemical and other properties. These changes are important in the practice therefore scientists have an urgent task to study and understand the mechanism of their emergence.

Fig. 1. Overview over different simulation tools and associated length scales and time scales.

Classical mechanics is based on a continuum description of material domains described in terms of empirical parameters. Continuum mechanics methods such as the finite element method (FEM) and the extended finite element method (XFEM) provide an economical numerical representation of material behavior at length scales in which continuum assumptions apply.
XFEM (and also certain meshfree methods) have been extended to multiscale modelling of cracks that bridge the meso and macroscale, see Fig. 1. These methods were applied to composite materials in order to gain a better understanding of these materials and in order to support their design. For example fibre–matrix separation and cracks in the matrix occur at much smaller scales than failure of the structure. Extension of XFEM to problems involving large deformation and fracture needs to be performed. This will require a consistent adaptation of material mass and voids over the elements through which cracks propagate. Significant theoretical and computational research is required to achieve predictive multiscale simulation tools based on XFEM which would be capable of modelling material failure from the onset of microcracking to global structural failure.

Molecular dynamics (MD) is a powerful tool for modelling of fracture in material with various internal structure and defects distribution. Methods of molecular dynamics are based on Lennard-Jones type pairwise potentials of interaction. The concept of pairwise potential implies that interaction of two particles depends only on their relative positioning and does not depend on position of any other particles. All potentials of Lennard-Jones type describe interaction for which pushing away at small distances and attracting at greater distances are characteristic. In the MD the big ensemble of the particles modeling atoms or molecules is considered. The particles interact with each other and besides can be subjected to external influence. In overwhelming majority of cases particles are considered as point masses. For all particles the equations of motion are formulated. The received system of the equations is solved by numerical methods. Rapid development of computer facilities gives possibility to solve such problems for the increasing number of particles. However, now it is impossible to model a macroscopic body, comparing each atom a point mass.

Atomistic models are well suited to gain a better understanding of material failure. However, atomistic simulations of dimensions relevant to engineering applications are prohibitively expensive. Therefore, one major research direction has been to develop multiscale methods that link atomistic domains to continuum (either meso- or macroscale) domains. The coupling can be achieved at discrete interfaces or over a so-called handshake domain. It is believed that handshake coupling methods enable a more gradual transition from one domain to another, e.g. they are capable of removing spurious wave reflections inherent to discrete interface coupling methods. There are many different coupling methods available such as the early quasi-continuum method, Arlequin method, coupling of length scales, bridging domain coupling, discrete-to-continuum bridging, to name a few.

2. Problem formulation for layered shells

Consider a layered, circular cylindrical shell with length \( l \) and radius \( R \) (see Fig. 2). The shell can be divided into \( n \) ring segments. The symbol \( n \) denotes the number of total ring segments separated from the rest cylindrical shell by the sections where the thickness variations take place. Every \( j \)-th ring segment of shell has \( q \) layers. Each layer is isotropic.
with thickness $h_{ij}$, Young’s modulus $E_{ij}$, Poisson’s ratio $\nu_{ij}$, and mass density $\rho_{ij}$ as show in Fig. 2.

Let’s denote

$$\rho_{ij} = \rho_{1j} \cdot d_{ij}, \quad (1)$$

where $d_{ij}$ is a constant of proportionality, $d_{1j} = 1$ and similarly Young’s modulus for each layer

$$E_{ij} = E_{1j} \cdot e_{ij}, \quad (2)$$

where $e_{ij}$ is a constant of proportionality, $e_{1j} = 1$.

We will denote the thickness of each layer $h_{ij}$ by

$$h_{ij} = (z_{i+1j} - z_{ij})h_{1j}, \quad (3)$$

where $z_{ij}$ is a local coordinate of a layer with the thickness $h_j$ and $z_{1j} = 0$.

The mass of $j$-th ring segment will be equal

$$\rho_{1j}h_{1j} \sum_{i=1}^{q} d_{ij}(z_{i+1j} - z_{ij}). \quad (4)$$

For the $j$-th ring segment, the differential equation of balance of a circular cylindrical shell with axial symmetry can be described by the equations [1]

$$\frac{\partial N_{1j}}{\partial x} = 0, \quad (5)$$

$$\frac{\partial^2 M_j}{\partial x^2} - \frac{N_j}{R} = 0,$$

where for thin shells one has [1]:

$$N_{1j} = \int_{0}^{h_j} \sigma_{1j} \, dz, \quad (6)$$

$$N_j = \int_{0}^{h_j} \sigma_j \, dz,$$
\[ M_j = \int_0^{h_j} \sigma_{1j} z \, dz, \]

where

\[ \sigma_{1j} = \frac{E_j}{1 - \nu_j^2} \left( \varepsilon_1 + z\chi + \nu_j \varepsilon \right), \]
\[ \sigma_j = \frac{E_j}{1 - \nu_j^2} \left( \varepsilon + \nu_j (\varepsilon_1 + z\chi) \right). \]

Let’s denote \( u_j(x, t) \) as the axial displacement and \( w_j(x, t) \) as the radial displacement of the \( j \)-th ring segment, where \( t \) is time.

If for all ring segments \( \nu_{ij} = \nu \), by using (1)–(3) and [1]

\[ \varepsilon_{1j} = \frac{\partial u_j}{\partial x}, \]
\[ \varepsilon_j = \frac{w_j}{R}, \]
\[ \chi_j = -\frac{\partial^2 w_j}{\partial x^2}. \]

For layered shells the force \( N_j \) and bending moment \( M_j \) (6) can be written as

\[ N_j = b_j E_{1j} h_{1j} \frac{w}{R}, \]
\[ M_j = -\frac{E_{1j} h_{1j}^3}{12(1 - \nu^2)} \bar{a}_j \frac{\partial^2 w}{\partial x^2}, \]

where

\[ \bar{a}_j = 4 \sum_{i=1}^{q} e_{ij}(z_{i+1j}^3 - z_{ij}^3) - 3 \left( \sum_{i=1}^{q} e_{ij}(z_{i+1j}^2 - z_{ij}^2) \right)^2 / \sum_{i=1}^{q} e_{ij}(z_{i+1j} - z_{ij}), \]

and

\[ b_j = \sum_{i=1}^{q} e_{ij}(z_{i+1j} - z_{ij}). \]

3. Effect of a multiscale factor on the Young’s modulus of a cylindrical shell

The most obvious distinction between bodies having the nanosizes and usual bodies is the growth of a role of near-surface area. Interaction between molecules (atoms) on a surface differs from volume as they have no neighbors from outer side. It is reason of reorganization of surface area. In volume, block, bodies the contribution of this layer with macroscopic properties is smallest, and it is usually neglected. However when the sizes of a body become small, commensurable with molecular sizes (nanodimensional), influence of near-surface area becomes considerable, and properties of substance qualitatively change.
The following example shows it. Consider a layered, circular cylindrical shell with length $l$ and radius $R$, like in p. 2. We will distinguish 2 types of layers: internal and external ones with molecular sizes. So we will have 2 surface layers and $n - 2$ internal layers. All layers have thickness $h_i$. Let’s denote

$$E_2 = \alpha E_1,$$

where $\alpha$ is a constant of proportionality between Young’s modulus of surface layer $E_2$ and Young’s modulus of internal layer $E_1$ and each layer has Poisson’s ratio $\nu$.

In this case bending moment $M$ (8) can be written as

$$M = -\frac{E_1 h_1^3}{12(1-\nu^2)} \bar{a} \frac{\partial^2 w}{\partial x^2},$$

where

$$\bar{a} = 4(\alpha(1 + n^3 - (n - 1)^3) + (n - 1)^3 - 1) - \frac{3(2\alpha n + (n - 1)^2 - 1)^2}{2\alpha + n - 2}.$$

Usually in the theory of shells bending moment looks like [1]

$$M = -\frac{E_\infty h_1^3 n^3}{12(1-\nu^2)} \frac{\partial^2 w}{\partial x^2},$$

where $E_\infty$ is Young’s modulus of shell.

By comparing formulas (11) and (12), we find

$$\frac{E_\infty}{E_1} = \frac{\bar{a}}{n^3}. \quad (13)$$

The results of calculations by (13) regarding to the tube with different constants of proportionality between Young’s modulus of surface layer $E_2$ and Young’s modulus of internal layer $E_1$ are presented in Fig. 3 – Fig. 4.

The influence number of layers $n$ in the tube for different values of $\alpha$ is depicted in Fig. 3. Here Young’s modulus of surface layer $E_2$ is larger than Young’s modulus of internal layer $E_1$, thus here is $\alpha > 1$.

In Fig. 4 different curves corresponding to different values of $\alpha < 1$. Here Young’s modulus of surface layer $E_2$ is less than Young’s modulus of internal layer $E_1$.

From Fig. 3 – Fig. 4 we can see that when $n < 200$, Young’s modulus of shell $E_\infty$ significantly depends on Young’s modulus of surface layer $E_2$ and $E_\infty/E_1 \to 1$ if $n \to \infty$.

### 4. Concluding remarks

When the sizes of a body become small, commensurable with molecular sizes (nanodimensional), the influence of near-surface area becomes considerable, and properties of substance qualitatively change. Therefore at nano level it is more preferable to consider layered model of a material.

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Fig. 3. Dependence $E_\infty/E_1$ from number of layers $n$ in the tube for the cases $\alpha = 2; 1.5; 1.3; 1.2; 1.1$.

Fig. 4. Dependence $E_\infty/E_1$ from number of layers $n$ in the tube for the cases $\alpha = 0.9; 0.7; 0.5; 0.3; 0.1$.

References

Pultrusion process optimization of thick composite profiles

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Abstract. Pultrusion is a high performance technological process of manufacturing composite profiles of constant cross section. The present work describes a method of pultrusion optimization of thick composite profiles using mathematical simulation techniques. The results of optimal solutions of the objective on pultrusion of the rod with large diameter at the change of the die temperature conditions and change of material temperature at entering into the die are given.

Keywords: pultrusion, curing, simulation, residual stresses, heat conduction.

1. Introduction

Pultrusion is a high performance technological process of continuous manufacturing composite profiles of constant cross section based on fiber reinforcement and polymer resin, including impregnation of the reinforcement with resin, polymerization in the die and cutting into measured segments of the manufactured profile. The pultrusion process represents a pulling process of the continuous reinforcement (glass roving, tape) with the help of the pulling unit through the impregnation unit with thermoset resin, which then enters the heated die that determines the geometry of the product cross section in which polymerization of the resin takes place (Fig. 1). Fig. 1 introduces the following terms: 1 – creels with the reinforcement, 2 – the impregnation unit, 3 – folder, 4 – the die, 5 – control panel, 6 – the pulling unit, 7 – the cutting unit.

Fig. 1. Scheme of pultrusion process.

A lot of attention is paid to development of mathematical models of the pultrusion of the long-length composite parts [1]–[4]. The existing math models consider separated elements of such technological process and allow to control these elements during manufacturing of products with relatively small sectional area Currently application of structural composite parts with large cross section area is very widespread in aviation, railway transport, construction and other industries. Examples of such products are: electrically insulating 80 mm diameter rods used for high-voltage insulators; structural elements of
Pultrusion technology does not depend on human factor. Hence, the critical task is to develop a pultrusion mathematical model for each type of product which will specify the most optimal criteria of this technology in order to receive a high quality product with the minimal cost. The main pultrusion process parameters which influence the properties of a final product and capacity are: pulling speed, temperature conditions on the die, temperature of the material at the die entrance, geometry of the product, properties of the components. Current work is devoted to the description of mathematical models for calculating the distribution of the mode of deformation, temperature and degree of polymerization during the pultrusion of products of various sections.

2. Problem formulation

During pultrusion manufacturing of the composite products it is necessary to find optimal combination of process parameters in order to increase the setup capacity and reduce the product cost with simultaneous receiving the required properties. In general, statement of the mathematical optimization problem involves several stages. At the first stage of statement of the mathematical optimization problem the simulation model is created which considers influence of different factors. But it is impossible to take into account all the properties of the actual objects during creation of a simulation model. That’s why it is necessary to define the main parameters of the actual process which will be included in the mathematical model. After selection of the simulation model the mathematical model describing the process is constructed. To estimate the solutions received during mathematical simulation it is necessary to introduce the optimality criteria. Using the optimality criterion, limitations, the mathematical model describing the process, mathematical apparatus of optimization (optimization method) allows to find the optimal solution for the design model.

Mathematical model of pultrusion was developed. It simulates the temperature field distribution, degree of curing, stress-strain distribution.

2.1. The heat transfer and polymerization model

Temperature field distribution inside the profile is simulated by heat conduction equation. Ignoring heat conduction in lengthwise direction and replacing time derivative with $x$ coordinate derivative multiplied by rate of motion $V$, the equation will be:

$$C \rho V \frac{\partial T}{\partial x} = \lambda \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \chi W V \frac{\partial \psi}{\partial x},$$

$$V \frac{\partial \psi}{\partial x} = K_0 \exp \left( - \frac{E_A}{RT} \right) (1 - \psi)^n,$$

$$T|_{\Sigma_0} = T_0, \quad \lambda \left. \frac{\partial T}{\partial \eta} \right|_{\Sigma_1} = \gamma (T - \theta(x)),$$

where $C$ – heat capacity; $\rho$ – density; $V$ – pulling velocity, $W$ – reaction heat; $\lambda$ – thermal conductivity; $\psi$ – degree of polymerization; $\chi$ – resin volume; $T$ – absolute temperature; $T_0$ – rod temperature at inlet of die; $K_0$ – preexponential factor; $E_A$ – active energy of process activation; $R$ – gas constant; $n$ – total reaction order by reacting components;
Implicit numerical algorithm based on finite differences was applied to solve the equations of heat conduction and polymerization for the profiles of different geometries.

2.2. Stress-strain state model

Nonlinear viscoelastic anisotropic model is used as the constitutive relations for the material [5]

\[ \sigma_{ij} = \begin{cases} C_{ijkl}^r \cdot (\varepsilon_{kl} - \varepsilon_{kl}^E), & T \geq T_g(\psi) \\ C_{ijkl}^g (\varepsilon_{kl} - \varepsilon_{kl}^E) - (C_{ijkl}^g - C_{ijkl}^r) \cdot (\varepsilon_{kl} - \varepsilon_{kl}^E)|_{t=t_{vit}}, & T < T_g(\psi) \end{cases} \]

where \( t_{vit} \) – the time of last transition of material from high-elasticity state to solid, \( \varepsilon_{kl}^E = \varepsilon_{kl}^T + \varepsilon_{kl}^C \) – expansion strains, \( \varepsilon_{kl}^T \) – thermal strains:

\[ \varepsilon_{kl}^T = \int_0^t \alpha_{kl}(T, \psi) \frac{\partial T}{\partial t} \, dt, \]

\( \varepsilon_{kl}^C \) – chemical shrinkage strains:

\[ \varepsilon_{kl}^C = \int_0^t \beta_{kl}(T, \psi) \frac{\partial \psi}{\partial t} \, dt, \]

\( C_{ijkl} \) - tensor of relaxation modules:

\[ C_{ijkl}(t) = \begin{cases} 0, & X < X_{gel} \\ C_{ijkl}^\infty + \sum_{p=1}^P C_{ijkl}^p \cdot \left( e^{-t/\rho_{ijkl}^p} \right), & X \geq X_{gel} \end{cases} \]

To implement this constitutive relation in finite element code the procedure of incrementation is used, i.e. output of the connection between increments of stresses and increments of deformations [5].

Boundary conditions for stress analysis are presented at the Fig. 2.

3. Results and discussion

In the present work the optimal solution is found when the temperature conditions are changed. Temperature conditions include temperature on the die and temperature of the material at the die entrance.
Let’s see how we can vary the temperature on the die. The die is heated with the heaters located in two rows. Software module of the pultrusion setup controls temperature of the heaters and temperature of the die in the heating area. The die can be divided into two zones. The first zone is located closer to the die entrance and the second zone is closer to the die exit. The first zone is heated with the first row of heaters and the second zone is heated with the second row of heaters. Thus, temperature on the die varied due to change of temperature in the first and second zones of the die. Temperature of the material at the die entrance may be increased by heating with the heating device.

According to the technical requirements to the setup the limitations are applied to the temperature conditions. For example, minimal temperature of 30°C at the die entrance corresponds to the conditions under which the preheating isn’t carried out, and maximal temperature of 70°C is maximal possible heating with using heaters included in the equipment. Minimal temperature of the first zone should be higher than 140°C, because at this temperature the resin gel formation begins with the subsequent polymerization. Increasing the temperature of the first zone the resin leads to more intensive heating of the resin and to acceleration of the gel formation and polymerization processes. But when temperature of the first zone is increased to 170°C small cracks occur on the part surface due to fast polymerization. So it is not recommended to increase the temperature of the first zone above 170°C. Temperature of the second zone is limited by 180°C because further temperature increase leads to material destruction.

In order to receive the required quality of the manufactured product the limitations were applied to the design values of the temperature field and degree of polymerization. The limitations were defined by the following technological criteria:

1. complete curing process during manufacturing of the part before going to the cutter;
2. absence of cracks in the part after exit from the die;
3. cooling down of the part before cutting;
4. absence of thermal destruction.

During the analysis of the design data the technological process was considered to be inapplicable if at least one of the criteria wasn’t satisfied. Thus, the mathematical model of the process included the simulation model, limitations for the model parameters and design data. Parameters of the mathematical model were considered to be optimal of the pullout speed reached the maximum.

Pultrusion of the 76 mm diameter rod was taken as an example of optimization of the pultrusion process. The optimal solution was found during changes of the material temperature at the die entrance and heating conditions of the material in the die. Two cases of variation of the temperature conditions were considered. In the first case the temperature on the die was recorded and the material temperature at the die entrance varied. In the second case both the temperature on the die and the material temperature at the die entrance varied. In the first case the inspection results of meeting the quality criteria are shown graphically, while in the second case the results are shown in cross tables. Six resin compositions have been analyzed (Table 1). Each resin composition had its own temperature conditions in which the pullout speed was maximal.

Parameters of the resin polymerization kinetics are shown in Table 2.

Below are the searching results of the optimal solutions to the problem of pulling out the rod of large diameter at changing material temperature at the die entrance. In process of solving the optimization problem we found the range of acceptable pullout speeds for six resin compositions. The problem was solved with the help of the software predicting the
pullout conditions under which high quality rods are manufactured. Temperature conditions on the die stayed the same (1-st zone – 140°С, 2-nd zone – 180°С).

Table 1. Resin compositions.

<table>
<thead>
<tr>
<th>Composition of the Resin</th>
<th>№ 1</th>
<th>№ 2</th>
<th>№ 3</th>
<th>№ 4</th>
<th>№ 5</th>
<th>№ 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Epoxy resin</td>
<td>100</td>
<td>100</td>
<td>70</td>
<td>70</td>
<td>70</td>
<td>70</td>
</tr>
<tr>
<td>Anhydride toughener</td>
<td>85</td>
<td>85</td>
<td>90</td>
<td>90</td>
<td>90</td>
<td>90</td>
</tr>
<tr>
<td>Epoxy-novolak resin</td>
<td>-</td>
<td>-</td>
<td>30</td>
<td>30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Cycloaliphatic epoxy resin</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Imidazole accelerator</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Dimethyldecylamine accelerator</td>
<td>1.8</td>
<td>-</td>
<td>1.8</td>
<td>-</td>
<td>1.8</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2. Parameters.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>№ 1</th>
<th>№ 2</th>
<th>№ 3</th>
<th>№ 4</th>
<th>№ 5</th>
<th>№ 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{tr}$, J/g</td>
<td>373.2</td>
<td>388.3</td>
<td>371.6</td>
<td>396.6</td>
<td>398.9</td>
<td>424.6</td>
</tr>
<tr>
<td>$n$</td>
<td>1.80</td>
<td>1.56</td>
<td>1.65</td>
<td>1.52</td>
<td>1.53</td>
<td>1.66</td>
</tr>
<tr>
<td>$E_a$, kJ/mol</td>
<td>127.2</td>
<td>116.9</td>
<td>126.1</td>
<td>113.6</td>
<td>112.3</td>
<td>114.3</td>
</tr>
<tr>
<td>$\ln K_0$</td>
<td>31.4</td>
<td>29.0</td>
<td>31.1</td>
<td>27.9</td>
<td>27.1</td>
<td>28.2</td>
</tr>
</tbody>
</table>

Fig. 3 provides an example of calculation of the range of acceptable pullout speeds for the resin № 6. The following designations are used in Fig. 3: 0 – all the criteria are satisfied, 1 – criterion № 1 is not satisfied, 2 – criterion № 2 is not satisfied, 3 – criterion № 3 is not satisfied, 4 – criterion № 4 is not satisfied.

Fig. 3. Suitable pulling speed depending on entrance temperature.

Analyzing the received results, we can draw the conclusions about the type of change of the range of acceptable pullout speeds depending on the material temperature at the die entrance. At low temperature (20–30°С) the maximum pullout speed doesn’t exceed 50 mm/min. As temperature increases, so does the maximum pullout speed. This is explained by the fact that at higher inlet temperature less time is needed to heat the material to the temperature of the polymerization onset, than at low inlet temperature. But
at the high material temperature at the die entrance and high pullout speed the part has no time to cool down (lower than the glass transition temperature) before going to the cutting table. Thus, further increase of the material temperature at the die entrance doesn’t lead to increase of the maximum pullout speed.

Comparative analysis of the pullout speeds of the products based on six types of resins shows that resins with imidazole accelerator have higher maximum pullout speed than resins with dimethyldodecylamine. The composition of the resin № 6 is more preferable. In this case the maximum pullout speed (70 mm/min) is reached at the minimal value of the inlet material temperature (50°C).

For example, if pulling speed is too high, quality criterion № 2 is failed. The crack is appeared (Fig. 4).

Simulated temperature distribution, degree of curing distribution, and transversal stress distribution are shown for Ø76 mm rod at Fig. 5.

![Crack](image)

**Fig. 4. Birth of crack.**

![Temperature, degree of curing, transversal stress](image)

**Fig. 5. Temperature, degree of curing, transversal stress for Ø76 mm rod.**

The proposed mathematical model of pultrusion optimization makes it possible to predict optimal combination of the technological parameters in order to increase the process
efficiency. This makes it possible to adjust the technological process in order to increase productivity and quality of the manufactured products.

References


The method of power series in constructing mathematical model of the dynamics of micropolar elastic thin bars

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Abstract. Dynamic equations, boundary and initial conditions of plane stress state of the micropolar theory of elasticity with independent fields of displacements and rotations are considered in thin rectangle. Using the method of expansion to power series along the thickness of rectangle and based on the initial approximation, applied one dimensional model of dynamic bending of micropolar elastic thin bars is constructed. It is shown that the constructed model coincides with the analagical model of micropolar bars, constructed on the basis of the asymptotically justified hypotheses method.

Keywords: power series method, model, micropolar bar.

1. Introduction

Main methods of reducing three-dimensional problem of theory of elasticity to two-dimensional or one-dimensional problem of applied theory of plates, shells and bars are divided into three main groups: (a) method of hypotheses, (b) method of expansion to power series along the thickness [1], c) asymptotic method. Asymptotic solution of three-dimensional boundary-value problem of micropolar theory of elasticity is constructed in paper [1]. Based on the qualitative aspects of the asymptotic solutions, adequate hypotheses are formulated and applied theory of the dynamics of micropolar elastic thin bars is constructed in paper [3]. In this paper method of expansion to power series along the thickness is developed to construct applied one-dimensional dynamic model of micropolar elastic bars and to compare it with the analagical model of bars, constructed in paper [3] on the basis of the asymptotically justified hypotheses method.

2. Problem formulation

A parallelepiped of constant height $2h$, length $a$ and constant thickness $2h_1$ is considered. The coordinate plane $x_1x_3$ is placed in the middle plane of the parallelepiped, which divides the thickness $2h_1$ in half. The axis $x_3$ is directed along the height and $x_1$-along the length of the parallelepiped. The axis $x_1$ divides the height $2h$ in half. We consider that plane stress state is realized in the parallelepiped in direction of axis $x_2$ and the problem of determining stress-strain state is reduced to a boundary value problem in the middle plane of the parallelepiped $x_1x_3$, that is, to study the problem in a rectangle $(0 \leq x_1 \leq a, -h \leq x_3 \leq h)$. We assume that $2h_1 = 1$. Basic equations of the dynamic plane problem of micropolar theory of elasticity with independent fields of displacements and rotations are followings [4]:

Motion equations:
\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{31}}{\partial x_3} = \rho \frac{\partial^2 V_1}{\partial t^2}, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho \frac{\partial^2 V_3}{\partial t^2}, \quad \frac{\partial \mu_{12}}{\partial x_1} + \frac{\partial \mu_{32}}{\partial x_3} - (\sigma_{13} - \sigma_{31}) = \int \frac{\partial^2 \omega_2}{\partial t^2}. \tag{1}
\]

Physical relations:
\[
\sigma_{11} = \frac{E}{1 - \nu^2} (\gamma_{11} + \nu \gamma_{33}), \quad \sigma_{33} = \frac{E}{1 - \nu^2} (\gamma_{33} + \nu \gamma_{11}),
\]
\[
\sigma_{13} = (\mu + \alpha) \gamma_{13} + (\mu - \alpha) \gamma_{31}, \quad \sigma_{31} = (\mu - \alpha) \gamma_{13} + (\mu + \alpha) \gamma_{31}, \quad \mu_{12} = B \chi_{12}, \quad \mu_{32} = B \chi_{32}. \tag{2}
\]

Geometrical relations:
\[
\gamma_{11} = \frac{\partial V_1}{\partial x_1}, \quad \gamma_{33} = \frac{\partial V_3}{\partial x_3}, \quad \gamma_{13} = \frac{\partial V_3}{\partial x_1} + \omega_2, \quad \gamma_{31} = \frac{\partial V_1}{\partial x_3} + \omega_2, \quad \chi_{12} = \frac{\partial \omega_2}{\partial x_1}, \quad \chi_{32} = \frac{\partial \omega_2}{\partial x_3}. \tag{3}
\]

Here \(\sigma_{11}, \sigma_{33}, \sigma_{31}, \sigma_{13}\) are power stresses; \(\mu_{12}, \mu_{32}\) are moment stresses; \(\gamma_{11}, \gamma_{33}, \gamma_{13}, \gamma_{31}\) are deformations; \(\chi_{12}, \chi_{32}\) are bending-torsions, \(V_1, V_3\) are linear displacements, \(\omega_2\) is free rotation of the points of rectangle around the axis \(x_2\); \(E, \nu, \alpha, B\) are the elastic constants of material of the micropolar body.

On the front lines \(x_3 = \pm h\) of rectangle power and moment boundary conditions (we shall discuss the problem of bending) are given:
\[
\sigma_{31} = p_1, \quad \sigma_{33} = \pm p_3, \quad \mu_{32} = \pm m_2, \text{ on } x_3 = \pm h \tag{4}
\]

Depending on how the application of external load is done or points are recorded, boundary conditions can be written down in power and moment stresses, in displacement and rotation or in mixed form.

It is assumed that the \(2h \ll a\) (i.e. the rectangle is thin).

### 2.1. The method of power series

For construction of one-dimensional model we apply method of bringing [1]. We approximate \(V_1, V_3\) and \(\omega_2\) to power series along the coordinate \(x_3\):
\[
V_1 = \sum_{n=0}^{\infty} V_{1,n} x_3^n, \quad V_3 = \sum_{n=0}^{\infty} V_{3,n} x_3^n, \quad \omega_2 = \sum_{n=0}^{\infty} \omega_{2,n} x_3^n. \tag{5}
\]
Substituting (5) into the above mentioned basic equations and boundary conditions of plane problem of micropolar theory of elasticity, taking place in a thin rectangle, we obtain the recurrence relations and conditions between the coefficients of the partial sums (polynomials) (5) of any degree, and the number of relations is equal to the number of unknown coefficients.

On the basis of (2), (3) with consideration of (5), for the power stresses \(\sigma_{33}, \sigma_{31}\) and moment stress \(\mu_{32}\) we obtain:
\[
\sigma_{33} = \frac{E}{1 - \nu^2} \sum_{n=0}^{\infty} [(n+1)V_{3,n+1} + \nu \omega_{1,n+1}] x_3^n, \tag{6}
\]
\[
\sigma_{31} = \sum_{n=0}^{\infty} \left[ (\mu - \alpha) \frac{\partial V_{3,n}}{\partial x_1} + (\mu + \alpha)(n + 1)V_{1,n+1} - 2\alpha \omega_{2,n} \right] x_3^n, \tag{7}
\]
\[
\mu_{32} = \sum_{n=0}^{\infty} B(n + 1) \omega_{2,n+1} x_3^n. \tag{8}
\]
Using formulas (6)–(8) and the boundary conditions (4) on the front lines \(x_3 = \pm h\) of the rectangle, after some transformations we obtain the following six equations:
\[
\sum_{k=0}^{\infty} \left[ (\mu - \alpha) \frac{\partial V_{3,2k}}{\partial x_1} + (\mu + \alpha)(2k + 1)V_{1,2k+1} - 2\alpha \omega_{2,2k} \right] h^{2k} = \frac{p_1^+ - p_1^-}{2}, \tag{9}
\]
\[
\sum_{k=0}^{\infty} \left[ (\mu - \alpha) \frac{\partial V_{3,2k+1}}{\partial x_1} + (\mu + \alpha)(2k + 2)V_{1,2k+2} - 2\alpha \omega_{2,2k+1} \right] h^{2k+1} = \frac{p_3^+ + p_3^-}{2}, \tag{10}
\]
\[
\frac{E}{1 - \nu^2} \sum_{k=0}^{\infty} \left[ (2k + 1)V_{3,2k+1} + \nu \frac{\partial V_{3,2k}}{\partial x_1} \right] h^{2k} = \frac{m_2^+ - m_2^-}{2}, \tag{11}
\]
\[
\frac{E}{1 - \nu^2} \sum_{k=0}^{\infty} \left[ (2k + 2)V_{3,2k+2} + \nu \frac{\partial V_{3,2k+1}}{\partial x_1} \right] h^{2k+1} = \frac{m_2^+ + m_2^-}{2}, \tag{12}
\]
\[
\sum_{k=0}^{\infty} B(2k + 1) \omega_{2,2k+1} h^{2k} = \frac{m_2^+ - m_2^-}{2}, \tag{13}
\]
\[
\sum_{k=0}^{\infty} B(2k + 2) \omega_{2,2k+2} h^{2k+1} = \frac{m_2^+ + m_2^-}{2}. \tag{14}
\]
Next, substituting the expression (3) into the generalized Hooke’s law (2), power stresses \(\sigma_{11}, \sigma_{13}\) and moment stress \(\mu_{12}\) will be expressed through displacement \(V_1, V_3\) and free rotation \(\omega_2\). Taking into consideration expansion (5), substituting into the motion equation (1), above obtained formulas for \(\sigma_{11}, \sigma_{13}\) and \(\mu_{12}\), we obtain following system of differential equations for the coefficients of the expansion (5):
\[
\frac{E}{1 - \nu^2} \left[ \frac{\partial^2 V_{1,n}}{\partial x_1^2} + \nu(n + 1) \frac{\partial V_{3,n+1}}{\partial x_1} \right] + (\mu - \alpha)(n + 1) \frac{\partial V_{3,n+1}}{\partial x_1} + \]
\[
+ (\mu + \alpha)(n + 2)(n + 1)V_{1,n+2} - 2\alpha(n + 1)\omega_{2,n+1} = \rho \frac{\partial^2 V_{1,n}}{\partial t^2}, \tag{15}
\]
\[
\frac{E}{1 - \nu^2} \left[ (n + 2)(n + 1)V_{3,n+2} + \nu(n + 1) \frac{\partial V_{3,n+1}}{\partial x_1} \right] + (\mu + \alpha) \frac{\partial^2 V_{3,n}}{\partial x_1^2} + \]
\[
+ (\mu - \alpha)(n + 1) \frac{\partial V_{1,n+1}}{\partial x_1} + 2\alpha \frac{\partial^2 \omega_{2,n}}{\partial x_1^2} = \rho \frac{\partial^2 V_{3,n}}{\partial t^2}, \tag{16}
\]
\[ B \frac{\partial^2 \omega_{2,n}}{\partial x_1^2} + B(n + 2)(n + 1) \omega_{2,n+2} - 2\alpha \left[ \frac{\partial V_{3,n}}{\partial x_1} - (n + 1)V_{1,n+1} + 2\omega_{2,n} \right] = \]
\[ = J \frac{\partial^2 \omega_{2,n}}{\partial t^2}. \]  
(17)

It should be noted that equations (15)–(17) and conditions (9)–(14) are split into two parts: symmetric with respect to \( x_3 \) (corresponding to longitudinal oscillations) and nonsymmetric (corresponding to the lateral oscillations).

On the basis of the above equations and conditions in the initial approximation of the power series transverse vibrations of micropolar thin rectangle (bar) are described by the following system of equations:

\[ (\mu - \alpha) \frac{\partial V_{3,0}}{\partial x_1} + (\mu + \alpha)V_{1,1} - 2\alpha \omega_{2,0} + \]
\[ + \left[ (\mu - \alpha) \frac{\partial V_{3,2}}{\partial x_1} + 3(\mu + \alpha)V_{1,3} - 2\alpha \omega_{2,2} \right] h^2 = \frac{p_1^+ - p_1^-}{2}, \]
\[ E \frac{1}{1 - \nu^2} \left[ 2V_{3,2} + \nu \frac{\partial V_{1,1}}{\partial x_1} \right] = \frac{p_3^+ + p_3^-}{2}, \]
\[ 2B \omega_{2,2} h = \frac{m_2^+ + m_2^-}{2}, \]
\[ \frac{E}{1 - \nu^2} \left[ \frac{\partial^2 V_{1,1}}{\partial x_1^2} + 2\nu \frac{\partial V_{3,2}}{\partial x_1} \right] + 2(\mu - \alpha) \frac{\partial V_{3,2}}{\partial x_1} + \]
\[ + 6(\mu + \alpha)V_{1,3} - 4\alpha \omega_{2,2} = \rho \frac{\partial^2 V_{1,1}}{\partial t^2}, \]
\[ \frac{E}{1 - \nu^2} \left[ 2V_{3,2} + \nu \frac{\partial V_{1,1}}{\partial x_1} \right] + (\mu + \alpha) \frac{\partial^2 V_{3,0}}{\partial x_1^2} + \]
\[ + (\mu - \alpha) \frac{\partial V_{1,1}}{\partial x_1} + 2\alpha \frac{\partial^2 \omega_{2,0}}{\partial x_1^2} = \rho \frac{\partial^2 V_{3,0}}{\partial t^2}, \]
\[ B \frac{\partial^2 \omega_{2,0}}{\partial x_1^2} + 2B \omega_{2,2} - 2\alpha \left[ \frac{\partial V_{3,0}}{\partial x_1} - V_{1,1} + 2\omega_{2,0} \right] = J \frac{\partial^2 \omega_{2,0}}{\partial t^2}. \]  
(18)

Substituting the second and third equations of (18) into the fifth and sixth equations, we obtain

\[ (\mu + \alpha) \frac{\partial V_{3,0}}{\partial x_1} + (\mu - \alpha) \frac{\partial V_{1,1}}{\partial x_1} + 2\alpha \frac{\partial^2 \omega_{2,0}}{\partial x_1^2} = \rho \frac{\partial^2 V_{3,0}}{\partial t^2} - \frac{p_3^+ + p_3^-}{2h}, \]  
(19)

\[ B \frac{\partial^2 \omega_{2,0}}{\partial x_1^2} + 2\alpha V_{1,1} - 2\alpha \frac{\partial V_{3,0}}{\partial x_1} - 4\alpha \omega_{2,0} = J \frac{\partial^2 \omega_{2,0}}{\partial t^2} - \frac{m_2^+ + m_2^-}{2h}. \]  
(20)

Substituting (5) into (2), for \( \sigma_{11} \) we obtain

\[ \sigma_{11} = \frac{E}{1 - \nu^2} \sum_{n=0}^{\infty} \left[ \nu(n + 1)V_{3,n+1} + \frac{\partial V_{1,n}}{\partial x_1} \right] x_3^n. \]  
(21)

In case of bending in the initial approximation we have
\[ \sigma_{11} = \frac{E}{1 - \nu^2} \left[ 2\nu V_{3,2} + \frac{\partial V_{1,1}}{\partial x_1} \right] x_3, \]  

\quad (22)

For the power stress \( \sigma_{31} \) at first we take

\[ \sigma_{31} = (\mu - \alpha) \frac{\partial V_{3,0}}{\partial x_1} + (\mu + \alpha)V_{1,1} - 2\alpha \omega_{2,0}. \]  

\quad (23)

Substituting expression \( \sigma_{11} \) (22) into motion equation (1), integrating by \( x_3 \), we obtain:

\[ \bar{\sigma}_{31} = \frac{x_3^2}{2} \left( \rho \frac{\partial^2 V_{1,1}}{\partial t^2} - \frac{E}{1 - \nu^2} \frac{\partial^2 V_{1,1}}{\partial x_1^2} - 2 \frac{E\nu}{1 - \nu^2} \frac{\partial V_{3,2}}{\partial x_1} \right) + \bar{\sigma}_{31}(x_1, t), \]  

\quad (24)

where \( \bar{\sigma}_{31}(x_1, t) \) is the constant of integration. For determination of this value, it is required that the averaged quantity of \( \bar{\sigma}_{31} \) along the height of the rectangle is equal to zero:

\[ \int_{-h}^{h} \bar{\sigma}_{31} \, dx_3 = 0. \]  

\quad (25)

Substituting (24) into (25), we obtain:

\[ \bar{\sigma}_{31}(x_1, t) = \frac{h^2}{6} \left( \frac{E}{1 - \nu^2} \frac{\partial^2 V_{1,1}}{\partial x_1^2} + 2 \frac{E\nu}{1 - \nu^2} \frac{\partial V_{3,2}}{\partial x_1} - \rho \frac{\partial^2 V_{1,1}}{\partial t^2} \right). \]  

\quad (26)

Thus, substituting (26) into (24), for \( \bar{\sigma}_{31} \) we obtain:

\[ \bar{\sigma}_{31} = \left( \frac{x_3^2}{2} - \frac{h^2}{6} \right) \left( \rho \frac{\partial^2 V_{1,1}}{\partial t^2} - \frac{E}{1 - \nu^2} \frac{\partial^2 V_{1,1}}{\partial x_1^2} - 2 \frac{E\nu}{1 - \nu^2} \frac{\partial V_{3,2}}{\partial x_1} \right). \]  

\quad (27)

So, we have:

\[ \sigma_{31} = 0 \sigma_{31} + \bar{\sigma}_{31}. \]  

\quad (28)

Substituting (23) and (27) into (28), we obtain the final formula for \( \sigma_{31} \):

\[ \sigma_{31} = (\mu - \alpha) \frac{\partial V_{3,0}}{\partial x_1} + (\mu + \alpha)V_{1,1} - 2\alpha \omega_{2,0} + \right) \left( \rho \frac{\partial^2 V_{1,1}}{\partial t^2} - \frac{E}{1 - \nu^2} \frac{\partial^2 V_{1,1}}{\partial x_1^2} - 2 \frac{E\nu}{1 - \nu^2} \frac{\partial V_{3,2}}{\partial x_1} \right), \]  

\quad (29)

where \( V_{3,2} \) is expressed by \( V_{1,1} \)

\[ \left( V_{3,2} = \frac{1 - \nu^2 p_3^+ + p_3^-}{2E} - \frac{\nu \partial V_{1,1}}{2 \partial x_1} \right). \]

Satisfying the appropriate boundary condition from (4), we obtain following equation from formula (29):

\[ (\mu - \alpha) \frac{\partial V_{3,0}}{\partial x_1} + (\mu + \alpha)V_{1,1} - 2\alpha \omega_{2,0} = \]  

\[ = \frac{p_3^+ - p_3^-}{2} - \rho \frac{h^2}{2} \frac{\partial^2 V_{1,1}}{\partial t^2}. \]  

\quad (30)

Combining (19), (20) and (30) definitively the following system of differential equations for the \( V_{3,0}, V_{1,1}, \omega_{2,0} \) will be obtained:
\[
(\mu - \alpha) \frac{\partial V_{3,0}}{\partial x_1} + (\mu + \alpha)V_{1,1} - 2\alpha \omega_{2,0} - \frac{h^2}{3} E \frac{\partial^2 V_{1,1}}{\partial x_1^2} - \nu \frac{h^2}{3} \frac{\partial}{\partial x_1} \frac{p_3^+ + p_3^-}{2h} = \\
= \frac{p_1^+ - p_1^-}{2} - \rho \frac{h^2}{3} \frac{\partial^2 V_{1,1}}{\partial t^2},
\]

(31)

\[
(\mu + \alpha) \frac{\partial^2 V_{3,0}}{\partial x_1^2} + (\mu - \alpha) \frac{\partial V_{1,1}}{\partial x_1} + 2\alpha \frac{\partial^2 \omega_{2,0}}{\partial x_1^2} = \rho \frac{\partial^2 V_{3,0}}{\partial t^2} - \frac{p_3^+ + p_3^-}{2h},
\]

\[
B \frac{\partial^2 \omega_{2,0}}{\partial x_1^2} + 2\alpha V_{1,1} - 2\alpha \frac{\partial V_{3,0}}{\partial x_1} - 4\alpha \omega_{2,0} = f \frac{\partial^2 \omega_{2,0}}{\partial t^2} - \frac{m_2^+ + m_2^-}{2h}.
\]

The system of equations (31) is the mathematical model of dynamics of micropolar elastic thin bars of bending deformation. It is a system of partial differential equations of hyperbolic type of the sixth order. Boundary conditions on the rectangle edges \((x_1 = 0, x_1 = a)\) [3] and the initial conditions for \(V_{3,0}, V_{1,1}, \omega_{2,0}, \partial V_{3,0}/\partial t, \partial V_{1,1}/\partial t, \partial \omega_{2,0}/\partial t\) should be attached to the system (31).

Now, we will compare the constructed model of the dynamics of micropolar elastic thin bars with analogical model [3], obtained on the basis of the asymptotically justified hypotheses method. We can see that the difference is only in underlined terms in (31). But this quantity is the result of the fact that power stress \(\sigma_{33}\) was kept in the physical relations for \(\gamma_{li} \ (i = 1, 2)\), which, as it is known, is neglected in the theory of bars.

References


Thin-walled cross-sections and their joints: tests and FEM-modelling

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Abstract. This paper is based on the experimental and numerical research of thin-walled cross-section’s compression resistance and shear strength of their joints carried out in St. Petersburg State Polytechnical University and HAMK University of Applied Sciences, Sheet Metal Centre. Current situation on the Russian market concerning the usage of cold-formed thin-walled cross-sections is aimed to find out a base foundation to start up a stipulation of the elements under discussion in the building industry. Some questions about the compression resistance of such cross-sections were raised at different conferences by scientific community and by companies such as Rautaruukki Oyj (Finland). Steel galvanized C- and H-profiles and thermo-profiles are types of thin-walled cross-sections that are normally used in small houses construction. Thermo-profiles have slots in webs that decrease the thermal flow through the web, but have a negative effect on strength of the profiles. These profiles were object of the research. Investigations carried out included tests to prove the compression resistance of the thin-walled cross-sections and shear strength of stud-to-rack joints. Numerical modeling of thin-walled cross-sections was done with contemporary analysis software (SCAD Office, Lira) using the finite element method (FEM).

Keywords: strength, buckling analysis, thin-walled cross-sections, joints.

1. Introduction

Studies undertaken by the authors in recent years have revealed that today’s building market in Russia is looking for building materials and technologies that could provide low-height housing industry with high-speed of construction, safety, ecological compatibility and finance efficiency.

The lightweight thin-walled cold-formed steel structures allow getting advantages that meet the requirements described above. Due to some reasons we, in Russia, do not have current norms that could be applied by engineers who design houses using the cold-formed steel structures. In this area a number of Ph D theses have been defended during recent years in Russia (G. I. Belyy, D. V. Kuzmenko, A. R. Tusnin, I. V. Astahov [1]). Theoretical research and laboratory tests were done only for specific types of thin-walled cross-sections.

Jyrki Kesti contributed a good deal to the development of local and distortional buckling of perforated steel wall studs [10]. Today thin-walled cold-formed steel structures won a good place in the Finnish building area. Experience that Finnish engineers have could help Russian science community to understand more exactly behavior of such a structures and appropriate European norms.
Summary of the research described below concerns reticular-stretched thermo-profiles. Reticular-stretched thermo-profile is a new type of thin-walled cross-sections that found its place in Russian market.

2. General

As object of research reticular-stretched thermo-profiles and their joints were analyzed (see Figure 1). The following profiles are discussed:

1. Specimen S1 (stud) - АИ ТСc 200-45-2,0;
2. Specimen S2 (rack) - АИ ПН 200-50-2,0.

Steel used for specimen production has the following parameters:

1. Steel grade S350GD (yield strength not less than 350 H/mm²);
2. Coating mass, 350 g/m²;

The research goal was to form the theoretical rationale for usage of reticular-stretched thermo-profile throughout buckling and shear strength analysis based on the laboratory tests.

Fig. 1. Reticular-stretched thermo-profiles.

Research tasks:

1. Laboratory tests:
   - Compression test;
   - Shear resistance test of stud-to-rack joint.
2. Numerical modeling (FEM):
   - Buckling analysis;
   - Shear resistance analysis of stud-to-rack joint.
3. Comparison of results.

Description and results of tests and numerical investigation are summarized below.
3. Experimental investigations

During experimental investigations the following tests were carried out:
- Compression test;
- Shear resistance test of stud-to-rack joint.

This paper describes parameters and results for some part of compression tests and shear resistance test of stud-to-rack joint. Tests were carried out in the Sheet Metal Centre at HAMK, using contemporary laboratory stand (Instron 3250), May 2013.

3.1. Compression test

Compression test parameters are described below (see Fig. 2, A).

1. Test specimen:
   - C–shaped thermo-slotted profiles АИ ТCc 200-45-2.0, web height 200 mm, flange width 45 mm, steel thickness 2.0 mm;
   - Total length of the test specimen 1000 mm;
   - The end of specimen are cut using a circular metal saw (the ends will not be machined);
   - Support blocks (thickness 40 mm; edge is positioned 3 mm from the end of the profile) made of wood are placed inside the profile at the ends.

2. Test arrangement:
   - The lower end of the specimen is placed on a hinged support made of steel (see Fig. 2, B);
   - The load of a hydraulic cylinder is applied through a thick steel plate to the upper end of the specimen.

3. Test procedure:
   - The specimen is loaded using the displacement control until the failure of the specimen;
   - The loading rate is 3 mm/min;

4. Test results:
   - Buckling force.

![Fig. 2. Compression test (A – specimen S3; B - hinged support).](image-url)
3.2. **Shear strength test of stud-to-rack joint**

Shear strength test parameters are described below. Drawing of the joint is presented below (see Fig. 3).

1. **Test specimen:**
   - Stud: the same profile type as in the compression test of length 350 mm;
   - Rack: AH IPH 200-50-2.0, U-shaped thermo-slotted profile of length 600 mm, web height 200 mm, flange width 50 mm, steel thickness 2.0 mm (the rack profile is press-braked on site and does not have thermal slots in web);
   - The flanges of the stud are fixed with 4+4 self-drilling screws ESSVE Wafer head screw ‘Non-Head’ Zinc drillpoint 4,8 × 16 to the flanges of the rack.

2. **Test arrangement:**
   - The stud is fixed to the head of the hydraulic cylinder;
   - The rack will be fixed rigidly to the test frame.

3. **Test procedure:**
   - The specimen is loaded using the displacement control until the failure of the specimen;
   - The loading is rate 1.5 mm/min

4. **Test results:**
   - Shear strength of the self-drilling screws.

![Fig. 3. Stud-to-rack joint. Scheme.](image)

![Fig. 4. Stud-to-rack joint. Clearance (app. 2.5 mm).](image)
3.3. Test results

Test results are shown on Fig. 7 – Fig. 9 and Table 1. Compression test were carried out with 4 specimen, but the specimen S3 is more correct describes behavior of the hinged profile (see Fig. 2). Resulting force that should be got out is buckling force. On the stress-strain diagram (see Fig. 7) one could see that buckling failure is achieved at app. 54 kN. Buckling mode is one half wave of the sinusoid. But anyway some strange behavior of the profile is shown before: short downfall of the current load. This stage demonstrates that firstly profile behave itself like with semi-rigid boundary conditions, than abrupt crash sound and profile behavior is like for the hinged one.
Fig. 7. Compression test diagram (S3).

Fig. 8 demonstrates common stress-strain diagram for self-drilling screws joint (specimen number C1) when the last one behave in shear. It should be said that real practice doesn’t allow for stud-to-rack joints have clearance between each member of the connection that one could see on Fig. 4. Only for the purpose of the shear strength test of stud-to-rack joint it was left small clearance (app. 2.5 mm). Each joint in shear behavior goes through the following stages:

- First stage (A–B) one could see like all the clearances (not the one that is described above) are got through;
- Second stage demonstrates elastic-plastic strain (B–C);
- Third stage (C–D) is noted by the yield segment of the diagram, strengthening, bearing failure of the sheet and achievement the ultimate strength (point D);
- Forth stage (after point D) - crushing of the joint. At the higher load level the end of the stud contacts the rack profile placed on a rigid base.

Fig. 8. Shear strength test diagram (C1).

Fig. 9 demonstrates the full diagram of the shear strength test where the ultimate strength is achieved at force that is equal to 93 kN.
Fig. 9. Shear strength test diagram (C2). Full edition.

Table 1. Tests results.

<table>
<thead>
<tr>
<th>Type of profile/ Specimen number</th>
<th>Shear strength</th>
<th>Compression test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ultimate breaking load, kN</td>
<td>Buckling force, kN</td>
</tr>
<tr>
<td>АИ ТСс 200-45-2,0 (S3)</td>
<td>-</td>
<td>53.82</td>
</tr>
<tr>
<td>АИ ТСс 200-45-2,0 (C1)</td>
<td>24.95</td>
<td>90.82</td>
</tr>
<tr>
<td>АИ ТСс 200-45-2,0 (C2)</td>
<td>23.93</td>
<td>92.99</td>
</tr>
<tr>
<td>АИ ТСс 200-45-2,0 (C3)</td>
<td>24.37</td>
<td>99.88</td>
</tr>
</tbody>
</table>

4. Numerical modeling (FEM)

Numerical modeling of thin-walled cross-sections and their stud-to-rack joint was done with contemporary analysis software (SCAD Office and Lira) using finite element method (FEM). FEM-models’ parameters were the same as for the tests described above. During the modeling process the thin-walled profile based on shell- and bar-elements and joint based on solid-elements were created and buckling/shear analysis tasks showed good results (see Fig. 10).

Fig. 10. FEM-model of thin-walled thermo-profile (A, shell-elements) and joint (B, solid-elements).
4.1. Buckling analysis

Numerical modelling and buckling analysis of the one meter thin-walled profile were carried out using FEM-software SCAD Office. FEM-models for buckling analysis have characteristics described below.

Quantitative characteristic of shell-element models:
- Number of nodes – 27485;
- Number of elements – 29519;
- Finite element dimensions – 3 mm.

Quantitative characteristic of bar-element models:
- Number of nodes – 11;
- Number of elements – 10;
- Finite element dimensions – 100 mm.

![Fig. 11. FEM-model. Strain-stress state of screws connection (A, B) and thin-walled profile (C).](image)

Point load (80 kN) was applied to the flexural center (FC) and points nearby it to justify different type profile deformation. It was shown that when load point is situated before FC (6.0, 9.0 mm from the outside surface of the web) thin-walled profile flexures inside itself (see Fig. 2, A and Fig. 11, C together) and difference between test and FEM-modelling results are equal to 0.33 and 3.75 % accordingly.

Other way profile behaves itself when the load point is situated after FC (11.5 mm from the outside surface of the web). This case turns up the deformation type (the thin-walled profile flexure outside itself) that is different from the test result. That’s why difference between test and FEM-modelling results is more and getting up to 6.03 %. FEM buckling analysis results are shown in Table 2.

4.2. Shear strength analysis

Numerical modelling and shear strength analysis of the screw joint were carried out using FEM-software Lira.

FEM-model for shear strength analysis has characteristics described below.

Quantitative characteristic of solid-element model:
- Number of nodes – 94659;
- Number of elements – 62736;
- Finite element dimensions – 3 mm.
As for screw elastic connection, it was decided to model it using two-node FE with unilateral elastic constraint between nodes.

Point load was applied to the flexural center (FC) step by step (5.0, 10.0, 15.0, 20.0, 25.0, 30.0 kN).

Real bearing failure the screw connection takes place due to crush of the element material under load that is equal to 25 kN (see Fig. 12). One could see on the FEM-model that main stress of steel nearby connection place is getting up to 335 MPa (more than yield strength of the steel, accordingly to Table 3.1b Eurocode 3 Part 1–3). Thereby large plastic deformations happen and round-form screw hole changes to oviform (see Fig. 13).

FEM shear strength analysis results are shown in Table 2.

5. Conclusions

Results of experimental investigation into the behavior of thin-walled cross-sections by compression (buckling analysis) and shear strength of their joints have been reported. For
both tests numerical analysis was carried out including bar/shell finite elements for compression and solid finite elements for shear strength analysis.

Table 2. FEM analysis results.

<table>
<thead>
<tr>
<th>Type of profile/Specimen number</th>
<th>Shear strength test</th>
<th>Shear strength analysis</th>
<th>Compression test</th>
<th>Buckling analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Solid</td>
<td>Shell</td>
<td>Bar</td>
</tr>
<tr>
<td></td>
<td>Ultimate breaking load, kN</td>
<td>53.82</td>
<td>55.83</td>
<td>6,83</td>
</tr>
<tr>
<td>AI TCc 200-45-2,0 (S3)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>AI TCc 200-45-2,0 (C1)</td>
<td>24.95</td>
<td>25.0</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Compression bar buckling has resulted in the axial failure of profile specimens S3 at a buckling force 53.82 kN. Results of numerical analysis (shell finite elements) differ from compression tests by about 4%. Bar finite element analysis show a little bit worse results that differ with tests in about 22%. The analysis clearly demonstrated that existing design guidelines for thin-walled cross-section modeling by bar finite elements are not exact and could be used only taking into account extra safety factor – 1,2.

Bearing capacity of the screw connection C1 (four screws on each profile flange) is equal to 24.95 kN and arithmetic mean value for three specimens (C1…C3) is about 24.42 kN. Real bearing failure the screw connection takes place due to crush of the element material. Results of numerical analysis (solid finite elements) differ from shear strength tests by about 2%. Summary of the investigations should be taken as a step to apply finite element method for modeling profile behavior without real tests.

Acknowledgements

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References


Optimal thickness of a cylindrical shell

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Abstract. In this paper an optimization problem for a cylindrical shell is discussed. The aim is to look for an optimal thickness of a shell to minimize the deformation under an applied external force with the volume staying constant during optimization. A model for the deflection is given and analyzed, followed by the formulation of an optimal control problem. Numerical solutions are presented for different examples.

Keywords: optimal control of PDE, shape optimization, linear elasticity.

1. Introduction

In this paper we discuss an optimization problem in linear elasticity, particularly in shape optimization. In this field, much research has been done in the last years, see e.g. [1]–[3]. The aim is to look for an optimal thickness of a cylindrical shell to minimize the deformation under an applied external force. As an additional restriction, the volume of the shell has to stay constant during the optimization process. The deflection is modelled using the so called “basic shell model” from Chapelle and Bathe [4], which makes use of the Hypothesis from Mindlin and Reissner. As a main result it is shown, that the resulting control-to-state operator is continuous. Followed by this, a corresponding optimal control problem is formulated and numerical solutions using FEM and standard nonlinear programming algorithms are presented for different examples.

2. Geometrical description of the shell

For the geometrical description, we first need a chart describing the midsurface of the shell. Let \( \omega \subset \mathbb{R}^2 \) be open and connected and \( \varphi: \bar{\omega} \to \mathbb{R}^3 \) be an smooth and injective mapping. We call \( S = \varphi(\bar{\omega}) \) the midsurface of the shell. We assume that the vectors \( a_\alpha := \partial \varphi / \partial \xi^\alpha, \alpha = 1, 2 \) are linearly independent and additionally consider an orthonormal vector

\[
a_3 := \frac{a_1 \times a_2}{\|a_1 \times a_2\|}.
\]

We call \( a_1, a_2 \), a covariant basis of the tangent plane of the midsurface and denote the corresponding contravariant basis by \( a^1, a^2 \).

We introduce \( t: \omega \to \mathbb{R}^+, t \in L^\infty(\omega) \) as the thickness of the shell and define the 3D-reference domain

\[
\Omega(t) = \left\{ (\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3 | (\xi^1, \xi^2) \in \omega, \xi^3 \in \left[ -\frac{t(\xi^1, \xi^2)}{2}, \frac{t(\xi^1, \xi^2)}{2} \right] \right\}
\]

together with the mapping

\[
\Phi(t): \Omega(t) \to \mathbb{R}^3, \Phi(t)(\xi^1, \xi^2, \xi^3) = \varphi(\xi^1, \xi^2) + \xi^3 a_3.
\]
Note that $\Phi(t)$ depends on the parameter $t$ only via its domain, but not in the right hand side. So the thickness parameter is supressed for $\Phi$ and the derived geometrical quantities in further text. We call $B(t) = \Phi(\Omega(t))$ the shell body. Let us denote the local covariant and contravariant basis with $g_l$ and $g^l$, and the components of the first fundamental form with $g^{ij} = g^i \cdot g^j$, $i, j = 1, 2, 3$. Furthermore we assume that $t(\xi^1, \xi^2) < 2|R_{\text{min}}(\xi^1, \xi^2)|$ with $R_{\text{min}}(\xi^1, \xi^2)$ being the radius of curvature of smallest modulus of the midsurface in $\varphi(\xi^1, \xi^2)$.

3. Modeling the displacement

We consider a small displacement $U: B(t) \to \mathbb{R}^3$ of the shell body. For modeling we use the Reissner-Mindlin kinematical assumptions, which state that normals to the midsurface remain straight and unstretched during deformation. This leads to the displacement approach

$$U(\xi^1, \xi^2, \xi^3) = u(\xi^1, \xi^2) + \xi^3 \theta(\xi^1, \xi^2)$$

with $u = u_1 a_1 + u_2 a_2 + u_3 a_3$ describing an infinitesimal displacement of all points on a line normal to the midsurface in $\varphi(\xi^1, \xi^2)$ and $\theta = \theta_1 a_1 + \theta_2 a_2$ representing a rotation vector.

We introduce the space of admissible displacements $V := \{(u, \theta)|u = u(u, \theta) \in H^1(S)^2 \times H^1(S), \theta \in H^1(S)^2\} \cap BC$ where $H^1(S)$ and $H^1(S)^2$ are Sobolev-spaces for scalar functions and first order tensors on the midsurface, respectively. Let us assume for the boundary conditions $BC$ that the shell body is hardclamped over the whole boundary $\partial S$, i.e. $(u, \theta) = 0$. We next consider the linear 3D-Green-Lagrange-strain tensor which is given by

$$e_{ij} = \frac{1}{2}(g_i \cdot U_j + g_j \cdot U_i), \quad i, j = 1, 2, 3,$$

where $U_i$ means the partial derivative of $U$ with regard to $\xi^i$. By Hooke’s Law, we get for the components of the stress tensor

$$\sigma^{ij} = \sum_{k,l=1}^{2} H^{ijkl} e_{kl}$$

with $H^{ijkl} = L_1 g^{ij} g^{kl} + L_2 (g^{ik} g^{jl} + g^{jl} g^{ik})$ and $L_1, L_2$ being the Lamé constants. Using the assumption that the normal stress $\sigma^{33}$ is zero this simplifies to

$$\sigma^{\alpha\beta} = \sum_{\lambda, \mu=1}^{2} C^{\alpha\beta\lambda\mu} e_{\lambda\mu}, \quad C^{\alpha\beta\lambda\mu} = \frac{E}{2(1 + \nu)} \left(g^{\alpha\lambda} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\lambda} + \frac{2\nu}{1 - \nu} g^{\alpha\beta} g^{\lambda\mu}\right),$$

$$\sigma^{3\alpha} = \sum_{\lambda=1}^{2} \frac{1}{2} D^{\alpha\lambda} e_{\lambda3}, \quad D^{\alpha\lambda} = \frac{2E}{1 + \nu} g^{\alpha\lambda}, \quad \alpha, \beta = 1, 2,$$

where $E$ is Young’s modulus and $\nu$ is Poisson’s ratio. We now consider a force $f \in L^2(S)$ and formulate the equilibrium conditions for the stationary case according to the basic shell model from Chapelle and Bathe [4]: Find $(u, \theta) \in V$ with
\[
\int \sum_{\alpha, \beta, \lambda, \mu=1}^2 C^{\alpha \beta \lambda} e_{\alpha \beta} (u, \theta) e_{\lambda \mu} (v, \psi) + D^{\alpha \lambda} e_{\alpha \lambda} (u, \theta) e_{\lambda \lambda} (v, \psi) \, dV = \int_{\omega} f \cdot v \, dS \quad (1)
\]

for all \((v, \psi) \in \mathcal{V}\). We define the bilinear form \(A(t)(u, \theta; v, \psi)\) for the left hand side and the linear form \(F(v, \psi)\) for the right hand side of (1).

4. Analysis of the problem

We know from [4] that \(A(t)\) is coercive and continuous for fixed \(t\), as well as \(F\) is continuous. According to the Lax-Milgram-Lemma there is an unique solution to (1). Therefore the control-to-state operator \(G\) which maps the control \(t\) to the corresponding displacement \((u, \theta)\) is well-defined. Let us define the set \(U_{\text{ad}} := \{t \in L^\infty(\mathcal{S}) | 0 < t_{\min} \leq t(\xi^1, \xi^2) \leq t_{\max} \text{ a.e. in } \mathcal{S}\}\). We now want to investigate the continuity of \(G: U_{\text{ad}} \to \mathcal{V}\).

**Lemma 1.** For all \(t \in U_{\text{ad}}\) the bilinear forms \(A(t)\) have a common coercivity constant, i.e.

\[
A(t)(u, \theta; u, \theta) \geq c\|\varphi(u, \theta)\|_{\mathcal{V}}^2 \quad \forall (u, \theta) \in \mathcal{V}.
\]

**Proof.** The proof can easily be derived from the original proof of coercivity in [4].

We next consider a sequence \(t_n \in U_{\text{ad}}\) that converges to \(\bar{t} \in U_{\text{ad}}\) and denote the corresponding sequence of states by \(y_n := (u_n, \theta_n) := G(t_n)\). Using the above Lemma we see that

\[
c\|y_n\|_{\mathcal{V}}^2 \leq A(t_n)(u_n, \theta_n; u_n, \theta_n) = F(u_n, \theta_n) \leq \|f\|_{L^2(\mathcal{S})}\|y_n\|_{\mathcal{V}},
\]

i.e. the sequence \(y_n\) is bounded. Hence there is a weakly convergent subsequence, also denoted by \(y_n\), with weak limit \(\bar{y} \in \mathcal{V}\).

**Lemma 2.** The weak limit \(\bar{y}\) is solution of (1) with thickness \(\bar{t}\).

**Proof.** We first find an alternative form for \(A(t)\), namely

\[
A(t)(u, \theta; v, \psi) = \int_{\Omega(t_{\max})} \sum_{i,j,k,l=1}^3 H^{ijkl} e_{ij}(u, \theta) e_{kl}(v, \psi) \chi(t) \, dV
\]

with

\[
\chi(t)(\xi^1, \xi^2, \xi^3) := \begin{cases} 1, & \text{if } -t(\xi^1, \xi^2)/2 < \xi^3 < t(\xi^1, \xi^2)/2, \\ 0, & \text{otherwise} \end{cases}
\]

This can be done since the integrand does not explicitly depend on \(t\) and \(\chi(t)\) is the characteristic function for \(\Omega(t)\). Another expression in terms of the \(L^2(B(t_{\max}))\) scalar product for second order tensors is

\[
A(t)(u, \theta; v, \psi) = \langle \sigma(v, \psi) \chi(t), e(u, \theta) \rangle_{L^2(B(t_{\max}))}.
\]

We have \(\sigma(v, \psi) \chi(t_n) \to \sigma(v, \psi) \chi(t)\) in \(L^2(B(t_{\max}))\) for fixed \((v, \psi) \in \mathcal{V}\), because
\[
\left\| \sigma(v, \psi) \chi(t_n) - \sigma(v, \psi) \chi(\bar{t}) \right\|^2_{L^2(B(t_{\text{max}}))} \leq \\
\sum_{i,j,k,l=1}^3 \int_{\Omega(t_{\text{max}})} g_{ijkl} \sigma^{ij}(v, \psi) \sigma^{kl}(v, \psi) \left( \chi(t_n) - \chi(\bar{t}) \right) dV.
\]

(2)

It holds \( \chi(t_n) - \chi(\bar{t}) \to 0 \) pointwise a.e., so we can conclude
\[
g_{ijkl} \sigma^{ij}(v, \psi) \sigma^{kl}(v, \psi) \left( \chi(t_n) - \chi(\bar{t}) \right) \to 0 \quad \text{pointwise a.e.}
\]

Furthermore \( \left| g_{ijkl} \sigma^{ij}(v, \psi) \sigma^{kl}(v, \psi) \chi(t_{\text{max}}) \right| \) is integrable majorant and we get the convergence of the right hand side from (2) to 0. From \( (u_n, \theta_n) \to (\bar{u}, \bar{\theta}) \) in \( V \) it follows that all components and covariant derivatives converge weakly to the corresponding limit in \( L^2(S) \), and so \( e(u_n, \theta_n) \to e(\bar{u}, \bar{\theta}) \) in \( L^2(B(t_{\text{max}})) \). We get
\[
\langle \sigma(v, \psi) \chi(t_n), e(u_n, \theta_n) \rangle_{L^2(B(t_{\text{max}}))} \to \langle \sigma(v, \psi) \chi(\bar{t}), e(\bar{u}, \bar{\theta}) \rangle_{L^2(B(t_{\text{max}}))}
\]

and therefore
\[
F(v, \psi) = \lim_{n \to \infty} A(t_n)(u_n, \theta_n; v, \psi) = A(\bar{t})(\bar{u}, \bar{\theta}; v, \psi).
\]

From the uniqueness of the limit \( \bar{u}, \bar{\theta} \) we conclude that the whole sequence converges weakly.

**Theorem 1.** The above convergence is also strong. Hence the operator \( G : U_{\text{ad}} \to V \) is continuous.

**Proof.** It holds for \( (v, \psi) \in V \)
\[
0 = A(t_n)(u_n - \bar{u}; \theta_n - \bar{\theta}; v, \psi) + A(\bar{t})(\bar{u}; \bar{\theta}; v, \psi) - A(\bar{t})(\bar{u}; \bar{\theta}; v, \psi).
\]

(3)

We now take \( (v, \psi) := (u_n - \bar{u}, \theta_n - \bar{\theta}) \) and have
\[
\lim_{n \to \infty} A(t_n)(\bar{u}; \bar{\theta}; u_n - \bar{u}; \theta_n - \bar{\theta}) - A(\bar{t})(\bar{u}; \bar{\theta}; u_n - \bar{u}; \theta_n - \bar{\theta}) = 0,
\]

because
\[
\left| A(t_n)(\bar{u}; \bar{\theta}; u_n - \bar{u}; \theta_n - \bar{\theta}) - A(\bar{t})(\bar{u}; \bar{\theta}; u_n - \bar{u}; \theta_n - \bar{\theta}) \right|
\leq \sum_{i,j,k,l=1}^3 \left| \left( e_{kl}(u_n - \bar{u}; \theta_n - \bar{\theta}), H^{ijkl} e_{ij}(\bar{u}; \bar{\theta}) \chi(t_n) - \chi(\bar{t}) \right) \right|_{L^2(B(t_{\text{max}}))}.
\]

(4)

Analog to the proof of the above Lemma we can show
\[
H^{ijkl} e_{ij}(u, \theta)(\chi(t_n) - \chi(\bar{t})) \to 0 \quad \text{in} \quad L^2(B(t_{\text{max}})),
\]
\[
e_{kl}(u_n - \bar{u}; \theta_n - \bar{\theta}) \to 0 \quad \text{in} \quad L^2(B(t_{\text{max}})).
\]

Both statements yield the convergence of the last term from (4) to 0. From (3) it follows
\[
0 = \lim_{n \to \infty} A(t_n)(u_n - \bar{u}; \theta_n - \bar{\theta}; u_n - \bar{u}; \theta_n - \bar{\theta}) \geq \lim_{n \to \infty} c \left\| (u_n - \bar{u}, \theta_n - \bar{\theta}) \right\|^2_V \geq 0
\]

and therefore the strong convergence \((u_n, \theta_n) \to (\bar{u}, \bar{\theta})\) in \( V \).  \( \square \)
5. Examples

We consider a part of a tube, also called panel. Optimal thickness problems for whole tubes and radially symmetric forces have been investigated e.g. in [3]. We take $\omega = (0,1) \times (0, \pi/2)$ together with $\phi(\xi^1, \xi^2) = (x, R \sin \phi, R \cos \phi)$ to describe the midsurface. We choose $E = 210, v = 0.3$ for the material parameters and $R = 1$ for the radius. For a fixed force $f = L^2(S)$ the optimization problem reads

$$\min_{t \in \Omega_{ad}} \| (u, \theta) \|^2$$

s.t. $A(u, \theta; \nu, \psi) = F(\nu, \psi)$, $\int_\omega t \, dS = C$.

We choose for $\| (u, \theta) \|$ an appropriate sum of $L^2(S)$ norms, $t_{\min} = 0.05, t_{\max} = 0.15$ and $C = 0.1 \pi/2$. The variational problem is solved using FEM and the optimization is done with Matlab's fmincon. The implementation follows ideas from [5] and uses general shell elements with nine nodes. We want to investigate two radially symmetric forces. The first one is $f^{(1)} = \xi^1 (1 - \xi^1)a_3$, the optimal thickness profile is shown in Fig. 1. We see that the thickness follows approximately the profile of the force (except at the boundary) and is in particular symmetric in both $\xi^1$ and $\xi^2$. Moreover it carries bang-bang characteristics. Our second example is an unsymmetric force $f^{(2)} = (\exp(\xi^1) - 1)a_3$. We see that the non-symmetry of the force is also represented in the optimal thickness profile shown in Fig. 2. We mention that the numerical results can be improved by using an exactly calculated gradient of the objective function and by evaluating necessary conditions for the optimal thickness as well as by adding a regularization term. The ”spikes” could be eliminated by using a finer mesh. A forthcoming paper will deal with these questions.

![Fig. 1. Optimal thickness profile to $f^{(1)} = \xi^1 (1 - \xi^1)a_3$.](image-url)
Fig. 2. Optimal thickness profile to $f^{(2)} = \exp(\xi^1) - 1)a_3$.

References


Short papers
Radiation defects the plastic deformed optical composite

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Development of processes of defect formation with education and stabilizing of \(H^-\) and \(I^-\)-interstitial with participation cation vacancies \((v_c)\) and bivacancy \((v_a,v_c)\) it is studied not enough. Meantime, exactly this initial stage of creation of stable nanosize defects anticipates passing to the terms, falling short of operating descriptions, and at the large doses of irradiation - and to mechanical destruction of optical composite. We nosed after the processes of creation of \(H^-\) interstitial of steady-state as trihalide molecules \(X3^-\) and interstitial loops interstitial single-crystals of \(KCl\), and also in plastic uniaxially of the deformed optical composite of \(KCl\) with the sharply enhanceable concentration of bivacency.

Analysis of experimental data for the plastic deformed optical composite. The analysis of experimental data shows for the plastic deformed optical composite of \(KCl\), that at \(\sim 37\) to part of \(I^-\) interstitial from three of \(F, I, V_K\) recombine with \(F^-\) centers, and the freed electron of radiate recombine with \(V_K^-\) - a center with characteristic luminescence of triplet autolocalized excition \((2,2\text{ eV})\). High intensity of \(I^-\) of peak of the thermostimulated luminescence for the deformed optical composite comports the sharp increase of number of \(F^-\) and \(V_K^-\) - of centers.

The optical compos of \(KCl\) with large anions and cations and not dense crystalline packing have record a high sensitiveness to influence \(X^-\) rays and electrons of subliminal energies. Done for \(KCl\) division of contributions to creation of defects from electron-hole recombine or disintegration of anionic eksitonov, and also the analysis of the united action of these mechanisms in the conditions of irradiation \(X^-\)rays will be allowed, to our opinion, to understand the mechanisms of functioning and other materials with \(E_{DF} < E_g\).
Physical-mechanical aspects creation radiation composites

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Mechanical properties of a composite are determined by relation of properties of reinforcing elements and matrix as well as bound strength between them. Effectiveness and work capacity of the material depend on correct selection of the initial components and technology of their combination, which shall provide stable bonds among components at keeping their initial characteristics.

Composite material films $d = 35$ microns and their composites doped with lavsan ($PET$) and polycarbonate ($MC$) have been investigated. Composite systems have been tested for tension (all-Union State Standard 14236-81 and 11262-80) at temperature $20 \pm 20^\circ C$, relative air moisture $45\%$ and moving speed of film holder $36.09$ mm/min. Holder moving connected to meter gauge has not exceeded $0.1$ mm.

Comparing the obtained results it is stated that $2\%$ $PET$ doping into polyimide leads to plasticity increase ($\sim 2.5$ times of the initial polyimide) and strength increase ($\sim 30\%$). However $PC$ doping leads to slight increase of relative elongation and decrease of strength of polymer composite. Spectrum analysis showed that $1\%$ $PC$ doping into polyimide leads to increase of spectrum intensity by $\sim 2 - 5$ times and significantly band width increases.

For $PET$ composites $5\%$ $PC$ doping into polyimide decreases spectrum intensity by $\sim 2 - 4$ times (with increase of irradiation dose) and washes away width of the existing bands, which connects to peculiarities of composite structure.

Composite materials destruction including irradiated by electrons composite materials (exponential, power series, linear) has been suggested based on calculation of deformation tensor and solution of integro-differential equation of balance, which describes the process of unaxial material destruction. Within the model the existing experimental data of relative elongation $\nu_3$ applied stress have been described based on polyimide, polyvinyliden fluorides, etc. It is obtained that for the most of polymer composite materials the experimental data of deformation $\epsilon \nu_3$ stress $\sigma$ are described well within the cascade-probability model.
Optimization of engineering structures using stochastic algorithms

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Keywords: global optimization, genetic algorithms, simulated annealing, optimization of truss, finite element method.

In the paper the methodology of mixed topology-shape-sizing optimization of several truss/frame engineering structures is presented. The objective function includes the price of structure. System of constraints comprises typical strength, stability and equilibrium conditions under the Eurocodes. Set of design parameters may consist of about 10 parameters of different nature. As the landscape of objective function possess many local minima, the global optimization algorithms (genetic algorithms, simulated annealing) are used. Numerical examples of optimization of tall guyed telecommunication masts and truss/frame bridges are presented.
Structural deformation and vibrational stability of radio telescope RT-32 with respect to the accuracy of astronomical measurements

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RT–32 radio telescope is fully-steerable 32 m parabolic antenna located in Irbene, Latvia. It is operated by the Engineering Research Institute “Ventspils International Radio Astronomy Center” (VIRAC) of the Ventspils University College, mainly for radio interferometric and radio astronomical research related projects.

Although the RT–32 is currently operational, there are some issues related to the efficiency and vibrational stability of the antenna mirror surface, discovered during the recent sessions of astronomical observations. The objective of ongoing studies is to characterize the response of backup structure of antenna to the static and dynamic loads in order to evaluate distortions of reflector surface and whole optical system. This paper summarizes results of modelling of deformations of the primary reflector system due to inertial and gravitational loads.

Finite Element Method (FEM) approach was used for stress-strain analysis. The digital model of RT–32 was developed as an assembly of beam and plate elements. Existing defects and wear of the construction (notably corrosion of steel tubes) were mapped by visual inspection and included in the model.

Different loading scenarios occurring during the operation of the telescope were considered (for example, rotation of the antenna with following rapid deceleration and bringing it to complete standstill).

Investigation of the vibrations of the construction during the repositioning of the reflector and its effect on the precision of astronomical observations is still work in progress.
After critical deformation multilayered plates of regular structure

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Keywords: post-critical deformation, multilayered plate.

This paper investigates the deformation after critical multi-layer plate consisting of the hard and soft layers of regular structure. Nonlinear strain components for expression as hard and soft layers produced using offset hypotheses for the conversion to the normal broken line. The expression for the total energy of the the layered plate. Applying the smoothing obtained equation’s variation.

The paper discusses after critical strain multilayer plate length of $2A$ and thickness of $2H$, consisting of hard and soft layers of $2b$ and thickness of $2h$ respectively exposed bilateral compression. Hard layers obey the Kirchhoff-Love hypothesis, and soft layers work for transverse shear and compression. Suitable nonlinear expression as hard and soft layers produced using offset hypotheses for the conversion to the normal broken line.

On the basis of the Kirchhoff-Love displacement hypothesis of any points of $i$-th layer of hard defined $\bar{w}_i(x, z), \bar{u}_i(x, z)$ by the formulas given V. V. Novozhylov [2]:

$$\bar{w}_i(x, z) = w_i(x) - \zeta_i(1 - \cos \theta_i), \quad \bar{u}_i(x, z) = u_i(x) - \zeta_i(1 - \sin \theta_i),$$  \hspace{1cm} (1)

where $\bar{w}_i(x), \bar{u}_i(x)$ - the vertical and horizontal displacements of the points of the middle surface, $\theta_i$ - rotation angle to normal, $-b \leq \zeta_i \leq b, i = 1, 2, ..., n$.

And for the deformation of this layer we have:

$$\bar{\varepsilon}_{xi} = \varepsilon_{xi} + \zeta \kappa_{xi},$$  \hspace{1cm} (2)

$$\varepsilon_{xi} = \frac{d u_i}{d x} + \frac{1}{2} \left(\frac{d w_i}{d x}\right)^2, \quad \kappa_{xi} = -\cos \theta_i \frac{d \theta_i}{d x} = - \frac{d \theta_i}{d s},$$  \hspace{1cm} (3)

where $\zeta \kappa_{xi}$ – arc length of the cylindrical surface, which addresses the median plane after deformation layer. Deformation (2) is negligibly small compared to unity, and although this is no limit on the angle of the normal rotation.

Applying the principle of smoothing, replacing finite sums of integrals $i$ over $z$ (ranging from 0 to $2H$), finite differences for the $i$-related differentials in $z$ and introducing the dimensionless variables $\xi = x/(2h_1); \eta = z/(2h_1)$ , arrive at the following variational equation:

$$\left[ -h \int_0^{H/h_1} (N_x + P) \delta u \delta \eta \right]_{\xi=0}^{\xi=a/h_1}$$
\[ -\left[ \int_0^{H/h_1} \left( N_x \frac{\partial w}{\partial \xi} + \frac{\partial M_x}{\partial \xi} + 4h_1^2 \tau_{xz} + 4h_1^2 \tau_{xz} \frac{\partial w}{\partial \eta} + \frac{2h_1 b}{h} \sigma_z \frac{\partial w}{\partial \xi} \right) \deltawd\eta \right]_{\xi=0}^{\xi=a/h_1} + \\
+ \left[ \int_0^{H/h_1} M_x \delta \left( \frac{\partial w}{\partial \xi} \right) d\eta \right]_{\xi=0}^{\xi=a/h_1} + \left[ 4h_1^2 \int_0^{H/h_1} \tau_{xz} \delta u d\xi \right]_{\eta=0}^{\eta=H/h_1} + \\
+ \left[ \int_0^{a/h_1} \int_0^{H/h_1} \left( \tau_{xz} \frac{\partial w}{\partial \xi} + \frac{2h_1^2 \sigma_z}{h} \frac{\partial w}{\partial \eta} + 4h_1^2 \sigma_z \right) \delta w d\xi d\eta \right]_{\eta=0}^{\eta=H/h_1} - \\
\left( \int_0^{a/h_1} \int_0^{a/h_1} \left( \frac{\partial N_x}{\partial \xi} + 2h_1 \frac{\partial \tau_{xz}}{\partial \eta} \right) \left( 2h_1 \delta u + \frac{\partial w}{\partial \xi} \right) d\xi d\eta - \\
- \int_0^{a/h_1} \int_0^{a/h_1} \left[ \frac{\partial^2 M_x}{\partial \xi^2} + 4h_1^2 \frac{\partial \tau_{xz}}{\partial \xi} + 4h_1^2 \frac{\partial \sigma_z}{\partial \eta} + N_x \frac{\partial^2 w}{\partial \xi^2} \\
+ 2h_1 \tau_{xz} \frac{\partial^2 w}{\partial \xi \partial \eta} + 2h_1 \frac{\partial}{\partial \xi} \left( \tau_{xz} \frac{\partial w}{\partial \eta} \right) + \frac{2h_1 b}{h} \frac{\partial}{\partial \xi} \left( \sigma_z \frac{\partial w}{\partial \xi} \right) \right] \delta w d\xi d\eta = 0. \]

References


The mathematical description of sustainability layered composites

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Keywords: multilayered wafer, composite, stability.

In this work received the equations of stability multilayered wafer consisting of hard and soft layers. Hard layers with bending stiffness, perceive marginal force as a soft layers serve as links among them and work for transverse shear and compression. Consider the effective method of solving the stability of laminated plates. The method is based on receiving the poise equations and boundary conditions of elastic stability of multilayered composite wafer on deformable substrates subject to bilateral compression in plane strain.

The problem is solved on the basis of the wafer laminated theory with the relations are not linear elasticity theory and the variation principle.

Following from the Kirchhoff-Love hypothesis leads to the following expression for the displacement components at any point of the first layer of ill:

\[ \ddot{u}_l(x, z) = u_l(x) - \zeta_l \frac{d w_l(x)}{dx}, \quad \ddot{w}_l(x, z) = w_l(x), \]  

where \( u_l(x), w_l(x) \) - the horizontal and vertical displacement of the points of the median plane; \( \zeta_l \) - the distance along the axis \( 0z \), measured from the plane, and \( -b \leq \zeta_l \leq b \).

To determine the transverse shear deformation and elongation of the normal soft layer also adopted assumption, it is introduced by A. Knight for the sandwich plates \cite{2}. It is this a straight element, normal to the median plane of the incompetent layer after the deformation or curl (stay straight, but not perpendicular to the middle surface). In this case, normal, originally drawn through all the layers of the plate, the deformation process is broken.

To derive the equations of elastic stability have used a variation principle that the second variation of the total energy \( \delta^2 \mathcal{E} \) of the system is taking to the state of “neutral” poise stationary value \cite{3}:

\[ \delta (\delta^2 \mathcal{E}) = 0. \]  

As a result, the equation of stability of laminated plates the finite differences for the displacement in terms of the deformations have been replaced by the principle of smoothing the respective differentials and approximate stability equations in partial derivatives. The work also assessed accuracy allowed by this substitution. Appropriate for this exact differential equations of elastic stability poise there by substituting the expression for system of the total energy. Expressing obtained through derivatives in the Taylor formula and substituting them in the mentioned differential equations, found values \( R_1 \) and \( R_2 \), make sense residual terms obtained approximate partial differential equation.

So, if the number of wafer layers \( r \) is sufficiently large, with high accuracy can be used poise equations of elastic stability.
References


Modern concept of fracture of composite materials

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In recent years, considerable interest in the kinetic theory of composites failure, based on the study of physical and mechanical processes caused by mechanical loads and exposure. The processes of deformation and fracture in polymers differ pronounced time dependence. This manifests itself in particular statistical phenomena of fatigue and creep, rupture stress decreases with increasing duration of action. A thorough study of these processes will allow the scientific approach to the creation of new high-strength polymers and method of protection from radiation damage.

It should be noted that among the most important problems of condensed matter physics is also the task of creating materials with predetermined physical properties. Require materials with high specific strength, heat-resistant, wear-resistant, resistant to corrosive environment, all kinds of radiation. One of the effective ways to solve this problem is the development of composites with different structures. Composites, unlike alloys and high-strength structural crystals successfully combine high strength with a greater toughness. To understand these processes need to take into account all the factors that affect the physical and mechanical properties of the composite: the nature of the polymer and filler phase and the physical state of the polymer, curing conditions and binding processes.

Phenomenological study of the mechanical properties of polymers in physics research continues to be reduced to the study of the rate of accumulation of violations (the emergence and growth of cracks) or return the integral value - load durability $\tau$. As above [2] according by Zhurkov formula durability of the applied voltage $\sigma$ and the temperature $T$:

$$
\tau = \tau_0 \exp \frac{U_0 - \gamma \sigma}{kT}
$$

is given the deep physical meaning and the further development the kinetic concept of strength is both on the basis the analysis of the physical meaning of this formula and its member coefficients $\tau_0$, $U_0$, $\gamma$ and on the basis of direct methods of studying the nature of the elementary events underlying the process of destruction.

In development of destruction questions, there are two basic approaches:
a) the destruction caused by invasion of the most dangerous defect when stress concentration coefficient reaches a critical value and
b) the destruction caused by the gradual accumulation of small defects in the bulk material.
The crack germination it is the final stage of destruction.

It should be noted that our knowledge about the true shape very approximate defect; significant role in the calculation averaged properties must play account viscoelastic
properties may therefore be more appropriate to determine the values of the coefficients $k_1, k_2$ of the experiment with a simple loading and further found using the prediction coefficients for changing the properties material in the other modes of loading.

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**References**


Optimization of ultrasonic flow meters for crude oil metering and export

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Keywords: Reynolds, Nusselt, ultrasonic meters, velocity profile, MATLAB, Microsoft Excel, CFD.

Ultrasonic flow meters, as all velocity or inference type devices, require an adequate flow stream conditioning in order to assure an accurate performance. Typical flow conditioning consists of straightening the upstream and downstream of the measuring section. The upstream section usually contains a tube bundle, which allows for the upstream section to be reduced in length. This tube bundle serves to eliminate any swirl in the flow stream before reaching the meter, presenting a symmetrical velocity profile to the turbine rotor. Some ultrasonic flow meters may produce a non-uniform pulse output, which can prove a wide span of repeatability. For such cases where is a need to correct the velocity flow profiles which affect the robustness of the integration method, this research work tries to develop a mathematical modeling and simulation in MATLAB and Microsoft Excel, with the purpose to combine the individual acoustic path measurements into a full volumetric flow rate measurement procedure. The relation between velocity and viscosity, using Reynolds Number, was calculated in Microsoft Excel. The Nusselt Number was then used to plot fluid mean temperature and wall temperature diagrams in Microsoft Excel.
Probability model of ship ageing

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The analysis of structures of ships has indicated that their strength and damage results from the phenomena of ships materials ageing. Ageing is characterized by cracks and corrosion. Assessment of ageing is best reflected by probability risk of ageing sensitivity. For its calculation it is necessary to know the density of damage in time and the function of ageing damage.
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